THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 1

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Before embarking on specific definitions, we first visit some familiar examples of bundles.

Example 1. Tangent Bundle of a Smooth Manifold.

Consider first a smooth m-dimensional submanifold, M, of $\mathbb{R}^n$. To each point $x \in m$ we can associate an m-dimensional linear subspace of the ambient $\mathbb{R}^n$. Denoted by T_xM, this is the *tangent plane to M at x*. Suppose that near x local coordinates on M are given by a map $G : \mathbb{R}^m \to \mathbb{R}^n$ (defined on a neighbourhood U of $0 \in \mathbb{R}^m$). Then T_xM is the m-plane through x and parallel to the image of $\mathbb{R}^m$ under the linearization of G at x, i.e. under the linear map defined by the Jacobian $[\frac{\partial G_i}{\partial x_j}(x)]$. Taking the union over all $x \in M$ we get the tangent bundle

$$TM = \bigcup_{x \in M} T_xM \ .$$

We can adapt this construction to the case of an abstract manifold of dimension m (rather than a submanifold of $\mathbb{R}^n$) if we identify vectors in T_xM with *directional derivatives at the point x*. The tangent space T_xM then acquires an interpretation as the vector space of *derivations on the germs of functions at $x \in M$*. If $\psi : U \to \mathbb{R}^m$ is a local coordinate chart defined on an open set $U \subset M$, and p is a point in U, then a basis for T_pM can be described as follows. Let $(x_1, \dots, x_m)$ be the coordinates on $\mathbb{R}^m$, and let $\psi(p) = x$. Define local vector fields $e_i(p) = \psi_*^{-1}(\frac{\partial}{\partial x_i}|_x)$ for $i = 1, \dots, m$. Then $\{e_1(p), \dots, e_m(p)\}$ is a basis for T_pM. As before, we define the tangent bundle TM by taking the union of the T_pM over all $p \in M$.

Notice the following features of TM:

(1): There is a projection map $\pi : TM \to M$ (defined by mapping T_pM to p) the fibers of which are m-planes, i.e. are copies of $\mathbb{R}^m$.

(2): the map $(p; v_1, \dots, v_m) \to \sum_{i=1}^m v_i e_i(p)$ defines an identification of $U \times \mathbb{R}^m$ with $TM|_U = \bigcup_{p \in U} T_pM$. (We say that TM is locally trivial)

The tangent bundle of a smooth manifold encodes information about the C^∞ structure of the manifold and is the essential construct for differential geometry.

Example 2. Normal Bundles. Suppose that $X \subset Y$ are smooth manifolds. For each $x \in X$, we have $T_xX \subset T_xY$ as a linear subspace. Define the normal at x to be the quotient,

$$N_x = T_xY/T_xX$$

If we have a metric/inner product defined, then

$$N_x = (T_xX)^\perp$$

i.e, N_x can be identified with the orthogonal complement to T_xM. Define the normal bundle to be

$$N = \bigcup_{x \in X} N_x$$

The normal bundle encodes information about how X sits inside of Y.

Example 3. The Hopf Map. Describe S^3, the standard 3-sphere as follows,

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \cong \mathbb{R}^4 : |z_1|^2 + |z_2|^2 = 1\}$$

There is a smooth identification of S^2, the standard two sphere, with complex projective space. That is, we may identify

$$S^2 \approx \mathbb{C}P^1 = \{[z_1, z_2] : 0 \neq (z_1, z_2) \in \mathbb{C}^2\}$$

where $[z_1, z_2]$ represents the equivalence class of (z_1, z_2) under the relation $(z_1, z_2) \sim (w_1, w_2)$ if and only if there is a $\lambda \in \mathbb{C}$ such that $(z_1, z_2) = \lambda(w_1, w_2)$. The identification is made using stereographic projection, say from the north pole $N \in S^2$, to identify $p_N : S^2 - \{N\} \to \mathbb{R}^2 \approx \mathbb{C}$. We can also identify $\mathbb{CP}^1 - [0, 1]$ with $\mathbb{C}$ via the map

$$[z_1, z_2] \mapsto \frac{z_2}{z_1}$$

The map that we want is

$$[z_1, z_2] \mapsto \begin{cases} p_N^{-1}(\frac{z_2}{z_1}) & z_1 \neq 0 \\ N & z_1 = 0 \end{cases}$$

Define $h : S^3 \to S^2$ by $(z_1, z_2) \mapsto [z_1, z_2]$. It is straight forward to show that this is a well defined, smooth surjection. The fiber over each point is

$$h^{-1}([z_1, z_2]) = \{\lambda(z_1, z_2) : \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1\} \cong S^1$$

Again, we have a sort of local triviality. Observe that $U_i = \{[z_0, z_1] : z_i \neq 0\}, i = 0, 1$ is an open set. Then, $h^{-1}(U_i) \cong U_i \times S^1$ via

$$(z_0, z) \mapsto ([z_0, z_1], \frac{z_i}{|z_i|}) = \begin{cases} ([1, \frac{z_1}{z_0}], \frac{z_0}{|z_0|}) & \text{over } U_0 \\ ([\frac{z_0}{z_1}, 1], \frac{z_1}{|z_1|}) & \text{over } U_1 \end{cases}$$

The inverse map on $U_0 \times S^1$ is given by

$$([1, z], \lambda) \mapsto (\frac{\lambda}{\sqrt{1 + |z|^2}}, \frac{\lambda z}{\sqrt{1 + |z|^2}})$$

with a similar definition on $U_1 \times S^1$. Observe that we do not have a globally trivial condition, since this would mean that $S^3 \cong S^2 \times S^1$. Using any number of invariants from algebraic topology (e.g, homology or homotopy groups) yields a contradiction.

Example 4. Tautological Bundles. Using the Hopf map $h : S^3 \to S^2$, we can get a new bundle, called a tautological bundle. If $(w_1, w_2) \in h^{-1}([z_1, z_2])$, then (w_1, w_2) is on the complex line defined by the pair $[z_1, z_2]$.

If we forget the restriction that $|w_1|^2 + |w_2|^2 = 1$, then we get a description of a complex line bundle over $\mathbb{C}P^1$. The notation is

$$\begin{array}{c} \vartheta_{\mathbb{C}P^1}(-1) \\ \Big\downarrow \pi \\ \mathbb{C}P^1 \end{array}$$

Where $\pi^{-1}([z_1, z_2])$ is the line defined by the pair $[z_1, z_2]$. This is called a tautological bundle. Note that we could perform the same construction for $\mathbb{CP}^n$.

Example 5. The Homogeneous Bundle. Let $O(n)$ be the set of $n \times n$ orthogonal real matrices. Observe that we can embed $O(n-1)$ into $O(n)$ by $Q_{n-1} \mapsto P_n$ where $P_n \in O(n)$ is the matrix with 1 in the upper left corner, zeros in the rest of the first row and column, and the Q_{n-1} in the remaining $(n-1) \times (n-1)$ square, i.e,

$$Q_{n-1} \mapsto \begin{pmatrix} 1 & * \\ * & Q_{n-1} \end{pmatrix}$$

Furthermore, $O(n)$ acts on S^{n-1} in the usual way via mulitplication,

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

The action is transitive. Set

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

The isotropy subgroup of e_1 is $O(n-1)$. The map,$O(n)/O(n-1) \to S^{n-1}$ given by $[A] \mapsto A \cdot e_1$ defines a diffeomorphism between S^{n-1} and the homogeneous space, $O(n)/O(n-1)$. Let $\pi : O(n) \to O(n)/O(n-1)$ be the natural map. For any $x \in O(n)/O(n-1)$, we have that $\pi^{-1}(x) = \{A \in O(n) : A \cdot e_1 = x\}$. Since $\pi^{-1}(e_1) = O(n-1)$, we get that $\pi^{-1}(x) = O(n-1)$. For, if $x = Q \cdot e_1$ for some Q, then $A \cdot e_1 = Q \cdot e_1$ implies that $Q^{-1}A \in O(n-1)$.

Exercise 1. Show that this bundle is locally trivial.

Remark. This last example generalizes to give bundles of the form

$$\begin{array}{c} G \\ \downarrow{\scriptstyle \pi} \\ G/H \end{array}$$

with fiber H, where G is any Lie group and $h \subset G$ is any closed subgroup.

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