THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 10

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Class Notes From Math 433

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February 13, 1998

Pull Back Constructions

Given a vector bundle and a smooth map f on manifolds:

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & B \end{array}$$

we can define a bundle $f^*(E) \to X$ by the requirement that the diagram commutes:

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\hat{f}} & E \\ {\scriptstyle p}\downarrow & \circ & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & B \end{array}$$

where $\hat{f}$ is an isomorphisms on fibers. Thus $f^*(E)|_x \cong E|_{f(x)}$.
<u>The construction:</u> Define the total space

$$f^*(E) = \{(x, e) \in X \times E | f(x) = \pi(e)\} \subset X \times E.$$

Then $p : f^*(E) \to X$ is given by $(x, e) \mapsto x$. So $p^{-1}(x) = \{(x, e)|\pi(e) = f(x)\} = E_{f(x)}$.

For local triviality of $f^*(E)$: Let $E|_{U_\alpha} \xrightarrow{\simeq^{\Psi}} U \times \mathbb{R}^n$ be the local trivialization and $V = f^{-1}(U) \subset X$.

Claim.

- $f^*(E)|_V \subset V \times E|_U$
- $f^*(E)|_V \cong V \times \mathbb{R}^n$(using Ψ)

Exercise 1. Prove the above claim.

Exercise 2. Show that if $E = B \times \mathbb{R}^n$, then $f^*(E) = X \times \mathbb{R}^n$.

For transition functions: If $\{V_\alpha\}$ is an open cover for X such that $\{U_\alpha = f(V_\alpha)\}$ is a cover of B and such that bundles are locally trivial, then the transition functions related by

$$g^f_{\alpha\beta}(x) = (f^* g_{\alpha\beta})(x) = g_{\alpha\beta}(f(x)).$$

Exercise 3. Prove the above equations.

Note. There is no corresponding construction for maps $f : B \to X$, that is, given

$$\begin{array}{ccc} E & & \\ \big\downarrow{\scriptstyle \pi} & & \\ B & \xrightarrow[f]{} & X \end{array}$$

we cannot define a bundle $f_*(E) \to X$ by $f_*(E)|_x = E|_{f^{-1}(x)}$ (for example, if f is not $1-1$, then the failure of this recipe is clear)!

Good Exercise 4 (Eugene Lerman). Consider

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\pi]{} & B \end{array}$$

Show that
(a) $\pi^*(E)$ is a bundle over B.
(b) $\pi^*(E) \cong E \oplus E$.

Exercise 5. Suppose that we have a bundle map between two bundles

$$\begin{array}{ccc} E' & \xrightarrow{\hat{f}} & E \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

such that $\hat{f}$ is an isomorphism on fibers. By the above construction applied to

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

we have a pullback $f^*(E)$.
Show that $f^*(E) \cong E'$.

Corollory. *$f^*(E)$ is uniquely defined!*

Theorem 1. (Key Result): *Suppose we have*

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle \pi} \\ X & \overset{f_0}{\underset{f_1}{\rightrightarrows}} & B \end{array}$$

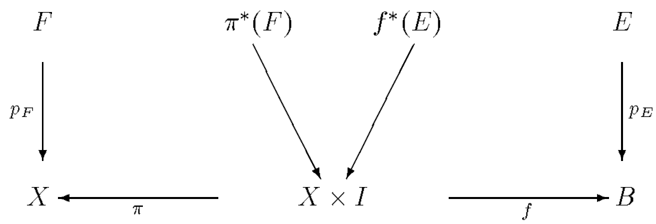
with $f_0 \simeq f_1$ (homotopic), then $f_0^*(E) \cong f_1^*(E)$.
(That is, homotopic maps produce isomorphic pullback bundles.)

Proof: (Bott & Tu) Suppose $f_t^*(E) \cong F$, for some f and $t \in [0,1]$.

Claim. *We can find* $I_\epsilon = (t-\epsilon, t+\epsilon)$ *such that* $f_s^*(t) \cong F$, $\forall s \in I_\epsilon$.

Then we can cover $[0,1]$ with finite number of intervals.
Hence $f_0^*(E) \cong F \cong f_1^*(E)$. $\square$

Proof of Claim: Look at bundles on $X \times I$:



Then we can get $\mathrm{Hom}(\pi^*(F), f^*(E))$ on $X \times I$ and $\mathrm{Iso}(\pi^*(F), f^*(E)) \subset \mathrm{Hom}(\pi^*(F), f^*(E))$.
If $f_t^*(E) \cong F$, then $\mathrm{Hom}(\pi^*(F), f^*(E))|_{X\times\{t\}}$ has a section, say,

$$\sigma : X \times \{t\} \to \mathrm{Iso}(\pi^*(F), f^*(E))|_{X\times\{t\}} \subset \mathrm{Hom}(\pi^*(F), f^*(E))|_{X\times\{t\}},$$

where $\mathrm{Hom}(\pi^*(F), f^*(E))|_{X\times\{t\}} = \mathrm{Hom}(\pi^*(F)|_{X\times\{t\}}, f^*(E)|_{X\times\{t\}})$.

To prove our claim, we must show

(a) σ extends to $X \times I_\epsilon$.

(b) The extension remains in Iso $\subset$ Hom.

For (a), we have the proposition:

Proposition. *For any vector bundle* $V \to X \times I$*, if* $\sigma : X \times \{t\} \to V|_{X\times\{t\}}$ *is a smooth section, then* σ *can extend to* $X \times I_\epsilon$*, for some* ϵ.

Proof: Temporarily omitted. $\square$

But then, (b) follows easily, since by continuity σ will remain in Iso on a suitably small strip I_ϵ. $\square$

Corollary. *If* B *is contractible, then* $E \to B$ *is trivial.*

Exercise 6. Using the above corollary to analogue bundles over S^n.

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