THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 11

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One of the central problems in vector bundles is the following classification problem. Given a space B and $n \in \mathbb{N}$, describe, up to isomorphism, all vector bundles of rank n over B. This is denoted by $\mathrm{Vect}_n(B)$.

Exercise 1. Using a two patch covering of S^1, examine the possibilities for the transition functions for a rank n bundle over S^1. Make a conjecture for $\mathrm{Vect}_n(S^1)$.

The idea for the classification is to find a bundle, denoted $\pi : EG \to BG$ and called the universal bundle for B, such that for any other bundle $E \to B$ of rank n, there is a map $f : B \to BG$ such that $f^*(EG) \cong E$. By our earlier results, we already know that homotopic maps induced isomorphic bundles. Thus, if $[B, BG]$ denotes the homotopy classes of maps from B to BG, then we have a well defined map,

$$\begin{aligned} [B, BG] &\to \mathrm{Vect}_n(B) \\ [f] &\mapsto [f^*(EG)] \end{aligned}$$

Our goal is to construct an inverse to this map. First, we have to say what EG and BG should be. We will assume throughout that B is compact, although the following will hold for when B is only paracompact.

Definition 1. A cover $\{U_\alpha\}$ of B is called a good cover if every non-empty intersection is diffeomorphic to $\mathbb{R}^d$, where d is the dimension of B. [Note: some texts call a cover good if every intersection is contractible.]

Note that for any bundle over B, we may take the elements of the good cover as our trivializing neighborhoods.

Proposition. *For $\pi : E \to B$ a vector bundle there exist finite many sections, $\{s_1, \ldots, s_k\}$ such that for all $b \in B$, the set $\{s_1(b), \ldots, s_k(b)\}$ spans the fiber.*

Proof: Consider first the local situation. Using the good cover, $\psi_i : E_{|U_i} \to U_i \times \mathbb{R}^n$, we claim that we have n local sections which generate E_b for all $b \in U_i$. If $\{e_1, \ldots, e_n\}$ is the standard basis, the local sections are (x, e_a). Define $s_{i,a} = \psi_i^{-1}(b, e_a)$. This is a local frame over U_i. To patch together to get a global set of sections, taker a partition of unity, $\{\rho_i\}$ subordinate to $\{U_i\}$. Extend $s_{i,a}(b)$ to $\rho_i(b)s_{i,a}(b)$. Then

$$\bigcup_i \{\rho_i s_{i,a}\}_{a=1}^n$$

do the job. $\square$

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Exercise 2. Check that the collection

$$\{\tilde{s}_i, \dots, \tilde{s}_k\} = \bigcup_i \{\rho_i s_{i,a}\}_{a=1}^n$$

globally generate the bundle E, i.e, $\{\tilde{s}_1(b), \dots, \tilde{s}_k(b)\}$ spans E_b for all $b \in B$.

Now that we have our global sections, how do we use them? Assume that $s_1, \dots, s_k$ are the desired sections. Let $V = \mathbb{R}\langle s_1, \dots, s_k\rangle$ be the real vector space on the set of sections. Fix $b \in B$ and define a map, $\mathrm{ev}_b : V \to E_b$ by evaluation, $s_i \mapsto s_i(b)$. Then, the following properties hold:

(a) The map is surjective since the $s_i(b)$ span the fiber.

(b) The kernel of ev_b is a codimension n subspace of V, so that $V/\ker(\mathrm{ev}_b) \cong E_b$

To complete our search for the universal bundle, we need to digress into the realm of the Grassmanian, denoted by $G(k, n)$, which is the set of all $k-$planes in $\mathbb{R}^n$. This is the subject of the next lecture.

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