THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 14

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Class Notes From Math 433

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Differential Geometry—Connections

Take

$$\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$$

smooth rank n vector bundle. ($\mathbb{C}$ or $\mathbb{R}$!)

Question: What is a connection on E?

Answer: There are three points of view.

(1) A connection is a device for computing derivatives of sections of a vector bundle.
(2) A connection is a device for decomposing tangent spaces to points in E into
   - Vertical directions (along fibers of of $E \to B$).
   - Horizontal directions ("parallel" to tangent directions to B).
(3) A connection is a device for comparing fibers of E at different points b_1 and b_2 by "parallel transport along a curve".

Remarks.

(1) The first point of view is easy to describe with respect to local frames.
(2) The second and third points of view can be applied to principal bundles as well as vector bundles.

1. Calculus for Sections Point of View

Denote the space of sections by $\Omega^0(B,E)$. Start with the trivial bundle $E = B \times \mathbb{R}^n$. Then we can think of $f \in \Omega^0(B,E)$ as $f : B \to \mathbb{R}^n$ (corresponding section is $b \mapsto (b, f(b))$) and for $f : B \to \mathbb{R}^n$ we can compute derivatives of f along a vector field on B using the differenial df of f.

If $n = 1$, so $f \in C^\infty(B,\mathbb{R})$, then df is a global 1-form on B, that is,

$$df \in \Omega^0(B, T^*B).$$

If $n > 1$, $f : B \to \mathbb{R}^n$, that is, $f(b) = (f_1(b), \cdots, f_n(b))$, then $df = (df_1, \cdots, df_n)$.

Exercise 1. Show that df is a global section of $T^*B \otimes \underline{\mathbb{R}^n}$, that is, $df = \sum \alpha_i \otimes s_i$, where $\alpha_i \in \Omega^0(B, T^*B)$ and $s_i \in \Omega^0(B, \underline{\mathbb{R}^n})$.

Key Properties of df

Think of the exterior derivative as a map

$$d : \Omega^0(B,\underline{\mathbb{R}^n}) \to \Omega^0(B, T^*B \otimes \underline{\mathbb{R}^n}) = \Omega^1(B, \underline{\mathbb{R}^n})$$
$$f \longmapsto df$$

Then

(1) d is linear.
(2) if $\lambda \in C^\infty(B, \mathbb{R})$, then $d(\lambda f) = d\lambda \otimes f + f d\lambda$ (Leibniz).

Conclusion: A connection can be thought of as a generalization of this map. For a non-trivial bundle E, a connection is defined to be a map

$$D : \Omega^0(B, E) \to \Omega^1(B, E)$$

such that

(1) D is linear.
(2) $D(\lambda s) = d\lambda \otimes s + \lambda Ds$, where $\lambda \in C^\infty(B, \mathbb{R})$ and $s \in \Omega^0(B, E)$.

2. Splitting of TE

Given a local trivialization $E|_U \cong U \times \mathbb{R}^n$, for any $e \in E|_U$, we have $T_eE = T_bU \oplus \mathbb{R}^n$, $b = \pi(e)$. This is local but not canonical, so we want global rule for splitting $T_e = Fiber \oplus Base$.

Fiber Directions: Given any vector $X_e \in T_eE$, $X_e \sim [\gamma(t)]$, where $\gamma(t)$ is a path through e at $t = 0$. Vectors "along the fiber $\pi^{-1}(b)$" correspond to paths $\gamma(t)$ which lie in $\pi^{-1}(b)$, that is, $\pi(\gamma(t)) = b$, $\forall t$. But the projection $\pi : E \to B$ induces $\pi_* : TE \to TB$ via $[\gamma(t)] \mapsto [\pi(\gamma(t))]$. So $\pi_*[\gamma(t)] = 0$, for paths $\gamma(t)$ which lie in the fiber $\pi^{-1}(b)$, since $\gamma(t)$ is a constant path at b. Therefore if we define $V_e \subset T_eE$ by $V_e = \mathrm{Ker}(\pi_* : T_e \to T_bB)$, then $V_e \cong \{\text{direction along the fiber}\}$.

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