THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 15

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Class Notes From Math 433

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February 25, 1998

Connections on Vector Bundles I

There are three points of view of connections:

(1) A connection is a device for computing derivatives of sections of a vector bundle.
(2) A connection is a device for decomposing tangent spaces to points in E into
- Vertical directions (along fibers of of $E \to B$).
- Horizontal directions ("parallel" to tangent directions to B).

(3) A connection is a device for comparing fibers of E at different points b_1 and b_2 by "parallel transport along a curve".

Before going on, we give a brief overview of how these points of view are related. We start with (1), that is, the problem of differentitating sections.

Given $f \in C^\infty(B, \mathbb{R})$, we get df, a global 1-form on B. That is,

$$d : C^\infty(B, \mathbb{R}) \to \Omega^1(B) = \Omega^0(B, T^*B)$$

such that $d : f \mapsto df$. Recall that $df_b : T_bB \to \mathbb{R}$ is given by $df_b(X_b) = X(f)(b) = \dfrac{d}{dt} f(\gamma(t))\Big|_{t=0}$ if $X_b \sim [\gamma(t)]$.

Now we can do the similar thing for $f : B \to \mathbb{R}^n$:

Given $X_b \sim [\gamma(t)]$, $\dfrac{d}{dt} f(\gamma(t))\Big|_{t=0} = (df_{1_b}(X_b), \cdots, df_{n_b}(X_b))$. So $df = (df_1, \cdots, df_n)$ measures variation of f along $\gamma(t)$, that is, $df \in \Omega^0(B, T^*B \otimes \underline{\mathbb{R}^n})$. Thus we get

$$d : C^\infty(B, \mathbb{R}^n) \to \Omega^0(B, T^*B \otimes \underline{\mathbb{R}^n}) = \Omega^1(B, \underline{\mathbb{R}^n}).$$

What about for $s : B \to E$, that is, $s \in \Omega^0(B, E)$?

How do we make sense of $\dfrac{d}{dt} s(\gamma(t))\Big|_{t=0}$?

Problem:

$$\left.\begin{array}{l} s(\gamma(t)) \in E_{\gamma(t)} \\ s(\gamma(0)) \in E_{\gamma(0)} \end{array}\right\} \quad \text{cannot be identified.}$$

Therefore we cannot evaluate $s(\gamma(t)) - s(\gamma(0))$.

If we had a way of "lining up"/ "identifying" all $E_{\gamma(t)}$ along $\gamma(t)$ (to $E_{\gamma(0)}$), then we could measure variation of s along $\gamma(t)$. Thus one way to solve the problem is by specifying how to transport E_b along $\gamma(t)$, that is, by defining **parallel transport**.

Without being too specific, to define parallel transports along a path in B we need to specify how paths in B should be lifted to a paths in E (The lifted paths are the so-called **horizontal lifts**). But this allows us to define a lifting of tangent vectors on B to tangent vectors on E. That is, we get an identification of T_bB with a subspace H_e of T_eE for any $e \in E_b$. The subspace H_e is a complementary summand in T_eE to the subspace of vertical vectors, that is, we get a splitting $T_eE = V_e \oplus H_e$.

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