THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 16

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Connections on Vector Bundles II

We now begin a detailed treatment of connections, starting with:

Definition 1. For a vector bundle $E \to B$, a *connection* is a map

$$D : \Omega^0(B, E) \to \Omega^0(B, T^*B \otimes E)$$

such that (i) D is linear and (ii) $D(fs) = df \otimes s + fDs$, for $f \in C^\infty(B, \mathbb{R})$ and $s \in \Omega^0(B, E)$.

Lemma. *(ii) guarantees that D is a local operator, that is, $Ds(b)$ depends only on s "near b".*

Proof: Take a open neighborhood U of b and define a smooth bump function f on U such that $f \equiv 1$ on $V(\subset U)$ near b and $f \equiv 0$ outside U. Then $D(fs) = df \otimes s + fDs$ implies $D(fs)(b) = Ds(b)$, but fs is zero outside U! □

Hence local descriptions of D are possible, that is, we can use local frames to get a local description of D. Say $E|_U \xrightarrow{\simeq^\Psi} U \times \mathbb{R}$ is a local trivialization, with corresponding local frame $\{e_i\}_{i=1}^n$. Thus $e_i(b) = \Psi^{-1}(b, \hat{e_i})$, for $\hat{e_i}$ a standard basis element for $\mathbb{R}^n$. Then, over U, with respect to this section, s has a local description $s = \sum s_i e_i$ with $s_i : U \to \mathbb{R}$. Applying (i) and (ii) in Definition 1, we can thus write $Ds = \sum ds_i \otimes e_i + s_i(De_i)$.

Claim.

$$De_i = \sum_j A_{ji} \otimes e_j$$

where A_{ji} are 1-forms defined on U.

Proof: De_i is a (local) section of $T^*U \otimes E$ and every section of $T^*U \otimes E$ has the form $\sum_j A_{ji} e_j$. (Note. A_{ji} depend only on D and $\{e_i\}$, NOT on s!)

Then

$$Ds = \sum_{i,j} (ds_j + A_{ji} s_j) \otimes e_i.$$

If we identify $s = (s_1, \cdots, s_n)^t$, we see that

$$D \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = (d + A) \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix},$$

where $A = [A_{ji}]$ is an $n \times n$ matrix of 1-forms on U. That is, $D = d + A$. □

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Note. If $E = B \times \mathbb{R}^n$, then only one patch is required, that is, we can pick $\{e_i\}$ to be a global frame. In this case, any matrix of 1-forms A will produce a connection $D = d + A$. In particular, $A \equiv 0$ yields $D = d$ (The difference between $D = d$ and $D = d + A$ will be clear later).

If $E = \coprod U_\alpha \times \mathbb{R}^n / \{g_{\alpha\beta}\}$ is not trivial and $\{e_i^\alpha\}$ is a local frame on U_α, then we must investigate compatibility of local descriptions:

Suppose $D = d + A^\alpha$ over U_α and $D = d + A^\beta$ over U_β, then

$$De_i^\alpha = A_{ji}^\alpha e_j^\alpha. \tag{1}$$

Also $e_i^\alpha = g_{ji}^{\alpha\beta} e_j^\beta$, so

$$\begin{aligned} De_i^\alpha &= D(g_{ji}^{\alpha\beta} e_j^\beta) \\ &= dg_{ji}^{\alpha\beta} \otimes e_j^\beta + g_{ji}^{\alpha\beta} De_j^\beta \\ &= dg_{ji}^{\alpha\beta} g_{kj}^{\beta\alpha} \otimes e_k^\alpha + g_{ji}^{\alpha\beta} A_{kj}^\beta g_{lk}^{\beta\alpha} \otimes e_l^\alpha \\ &= \left(\left[g^{\beta\alpha} \cdot dg^{\alpha\beta} \right]_{li} + \left[g^{\beta\alpha} A^\beta g^{\alpha\beta} \right]_{li} \right) \otimes e_l^\alpha. \end{aligned} \tag{2}$$

Therefore the compatibility implies (1)=(2). Hence

$$A^\alpha = g^{\beta\alpha} A^\beta g^{\alpha\beta} + g^{\beta\alpha} \cdot dg^{\alpha\beta}. \tag{$*$}$$

Conclusion: D is specified by $\{A^\alpha\}$, where A^α is a matrix value 1-form on U_α and A^α and A^β related by $(*)$ on $U_\alpha \cap U_\beta$.

Note. We cannot take $A^\alpha \equiv 0$, because this does not in general satisfy $(*)$.
Exception: If $dg^{\beta\alpha} = 0$, that is, $g^{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n)$ is (local) constant. For reason which will become clear, such bundles are called **flat bundles**:

Definition 2. If E admits description $E = \coprod U_\alpha \times \mathbb{R}^n / \{g_{\alpha\beta}\}$ such that $dg^{\alpha\beta} = 0$, then E is called *flat*.

If E is a flat bundle, we can define a connection $D = d$ with respect to corresponding flat local frames.

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