THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 17

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Existence of Connections

Say that a bundle is described by

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

Here are two ways to construct a connection on E. Both methods use connections, D_α defined on $E|U_\alpha$, the local trivializations $\psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^n$ and a partition of unity, $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. The two connections are defined by

(a) $D_1 = \sum_\alpha D_\alpha \circ \rho_\alpha$
(b) $D_2 = \sum_\alpha \rho_\alpha D_\alpha$

Note. D_α was defined on $E|U_\alpha$ by:

(a) Picking a connection, $\tilde{D}_\alpha$ on $U_\alpha \times \mathbb{R}^n$.
(b) Setting $D_\alpha = \psi_\alpha^{-1} \circ \tilde{D}_\alpha \circ \psi_\alpha$, i.e, $D_\alpha(s) = \psi_\alpha^{-1}(\tilde{D}_\alpha(\psi_\alpha(s)))$

Exercise 1. Show that this is a connection.

In fact, given a bundle isomorphism, $h : E \to F$ and a connection D on F, we define a connection, h^*D, on E by $h^*D = h^{-1} \circ D \circ h$.

In the definition of D_2, the terms in the sum are of the form

$$(\rho_\alpha D_\alpha(s))(b) = \rho_\alpha(b) D_\alpha(s)(b) = \begin{cases} \rho_\alpha(b) D_|\alpha(s)(b) & b \in U_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Exercise 2. Check that D_2 does define a connection.

We now examine the connection, D_1. We define

$$\begin{aligned}(D_\alpha \circ \rho_\alpha)(s) &= D_\alpha(\rho \cdot \alpha(s)) \\ &= d\rho_\alpha \otimes s + \rho_\alpha D_\alpha(s)\end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Using this, we see that

$$D_1(s) = D_2(s) + \sum_\alpha d\rho_\alpha \otimes s$$

[Note that off of U_α, $d\rho_\alpha = 0$ since the support of ρ_α is in U_α.]

Exercise 3. Show that D_1 defines a connection.

The diffrerence between D_1 and D_2 is instructive.

Note. $D_1 - D_2$ is a End(E) valued 1-form. In fact

$$(D_1 - D_2) = \sum_\alpha d\rho_\alpha \otimes \mathrm{Id}$$

$$\text{i.e, for any } s \in \Omega^0(E) \quad (\sum_\alpha d\rho_\alpha \otimes \mathrm{Id})(s) = \sum_\alpha d\rho_\alpha \otimes s$$

This illustrates a general fact.

Proposition. *If D_1 and D_2 are connections on a vector bundle, $\pi : E \to B$, then*

$$\begin{aligned} D_1 - D_2 &\in \Omega^0(B, T^*B \otimes End(E)) \\ &= \Omega^1(B, End(E)) \end{aligned}$$

Proof: We check:
(a) This is linear with respect to constants.
(b) This is also linear with repsect to functions:

$$\begin{aligned} (D_1 - D_2)(fs) &= (df \otimes s + fD_1(s)) - (df \otimes s + fD_2(s)) \\ &= f(D_1 - D_2)(s) \end{aligned}$$

Note. With respect to local frames of E, say $\{e_i^\alpha\}_i^{\text{rank E}}$, locally a section of $T^*M \otimes$ End(E) is a matrix of 1-forms, A^α. Over U_α, A^α and A^β must be related by the transition functions,

$$A^\alpha = g_{\alpha\beta} A^\beta g_{\beta\alpha}$$

If $(D_1 - D_2)(fs) = f(D_1 - D_2)(s)$, then the local descriptions (with respect to local frames) will also have this property.

Exercise 4. Repeat the computation in the proof of (b) for connection 1-forms.

Summary: $D_1 - D_2 \in \Omega^1(B, \mathrm{End}(E))$. Conversely, if $A \in \Omega^1(B, \mathrm{End}(E))$, then we can define $(D_1 + A)(s) = D_1(s) + A(s)$.

Exercise 5. Show that $D_1 + A$ is a connection.

Conclusion: If $\mathcal{A}(E)$ is the space of all connections on E, then $\mathcal{A}(E) = D_0 + \Omega^1(B, \mathrm{End}(E))$ where D_0 is any fixed connection. That is, $\mathcal{A}(E)$ is an infinite dimensional affinve space based on $\Omega^1(E, \mathrm{End}(E))$.

Parallel Transport

Definition 1. Given a connection, D, and a vector field, X, then $D_X s$ is called the *covariant derivative* of s along X. We can write it locally: $D_X s = d_X s + (A(X))s$, where $D_x s \in \Omega^0(B, E)$ and $(D_X s)(b) = (d_{X_b} s) + A(X_b)s(b)$, that is, it depends only on $X_b \in T_b B$.

Definition 2. If $s \in \Omega^0(B, E)$ satisfies $Ds = 0$, then we say that s is *parallel.*

Question. Can we find solutions to $Ds = 0$?

With respect to a local frame, if

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \cdots \\ s_n \end{pmatrix}$$

and $D = d + A_1$, then the condition for being parallel is $ds_i + A_{ij} s_j = 0$ for all i. With local coordinates, $(x_1, \ldots, x_k)$, on B, we are thus looking at trying to solve the system of equations,

$$\sum_{\alpha=1}^{k} \frac{\partial s_i}{\partial x_\alpha} + (A_{ij}^\alpha s_j) dx_\alpha = 0 \quad i = 1, \ldots, n$$

where $A_{ij} = A_{ij}^\alpha dx_\alpha$. This is a system of partial differential equations for which existence of solutions is NOT guarenteed.

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