THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 18

Professor Steven Bradlow
Class Notes From Math 433

University of Illinois at Urbana-Champaign

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Given a connection D on $E \to B$, a rank k bundle over a $n-$manifold, B, we defined a section, $s \in \Omega^0(B, E)$ to be parallel if $Ds = 0$. We saw that the existence of parallel sections, e.g, solutions to Ds=0, was tricky; i.e, it involved solving a set of partial differential equations. We can, however, look at a special case of the problem were a solution is possible.

Given a curve, γ, in B, we can look at the section along the curve, $s(\gamma(t))$. Recall that we get a vector field along γ, denoted by $\dot{\gamma}(t)$. This is the velocity vector field given by ,

$$\dot{\gamma}(t)(f) = \frac{\partial}{\partial \tau} f(\gamma(\tau))_{|\tau=t}$$

We can now examine, $D_{\dot{\gamma}}s$, the covariant derivative of s along $\gamma(t)$, i.e, study solutions to $D_{\dot{\gamma}}s(\gamma(t)) = 0$.

Claim. *This is (locally) a system of ordinary differential equations (first order linear).*

Proof: With respect to a frame, $\{e_i\}$ for E, we can write $D = d + A$ and

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \dots \\ s_k \end{pmatrix}$$

The equations that we are interested in now become $ds_i + A_{ij}s_j = 0$. So, $D_{\dot{\gamma}}s = 0$ becomes

$$d_{\dot{\gamma}(t)}s_i + A_{ij}(\gamma(t))s_j(\gamma(t)) = 0.$$

$$\text{But, } d_{\dot{\gamma}(t)}s_i = \dot{\gamma}(t)(s_i) = \frac{\partial}{\partial \tau} s_i(\gamma(\tau))_{|\tau=t}.$$

Now, call $s_i(\gamma(t)) = s_i(t)$ and $A_{ij}(\gamma(t)) = A_{ij}(t)$. So, we have

$$\frac{\partial}{\partial \tau} s_i(t) + A_{ij}(t)s_j(t) = 0, \ i = 1, \dots, k$$

A basic fact from a course in ordinary differential equations is that this has a unique solution if we specify initial conditions. That is, if we specify what $s_i(0) = s_i(\gamma(0)$ is. These solutions are valid wherever $\gamma(t)$ is defined (so long as the γ is contained in one coordinate patch!) If γ runs across several patches, then a slightly beefed up argument is needed. This is left as an exercise. Regardless, the solution is a curve in E over B and is called the horizontal lift of γ.

Note. We can always lift a path through B to E. However, we have specified a very special lifting, one which has certain parallel properties. We denote this distinguished horizontal lift by $\tilde{\gamma}_h$. This leads to the following definition.

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Definition 1. Horizontal lifting defines a map, $T_bB \to T_eE$ by $X \sim [\gamma(t)] \mapsto [\tilde{\gamma}_h(t)] \sim \tilde{X}_h$. $\tilde{X}_h$ is called the *horizontal lift of the vector* X. The image of this map is a dimension n subspace of T_eE, complementary to $V_e \subset E$. That is, the image defines the *horizontal subspace* H_e.

Parallel Transport

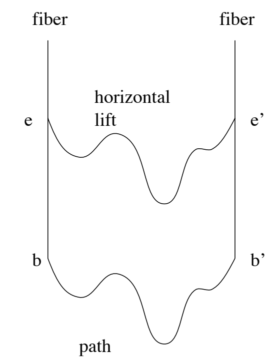

Figure 18.1

Suppose that $b, b' \in B$. We want a way to compare $E_b, E_{b'}$, the fibers over the repsective points. Let $\gamma : [0,1] \to B$ be a path such that $\gamma(0) = b$ and $\gamma(1) = b'$. Let $\tilde{\gamma}_h$ be the horizontal lift of γ such that $\tilde{\gamma}_h(0) = e$. Then, the comparison is just $e \mapsto \tilde{\gamma}_h(1)$. This is represented in figure 18.1.This most definitely depends on the choice of the path γ.

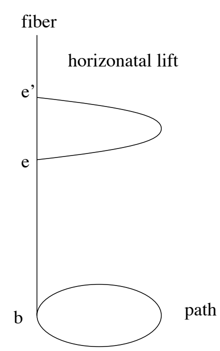

Figure 18.2

For example, if $b = b'$, then γ is a loop based at b. However, there is no guarentee that the horizontal lift, $\tilde{\gamma}_h$ will be a loop. This is illustrated in figure 18.2. In fact, $\tilde{\gamma}_h(1) = T \cdot \tilde{\gamma}_h(0)$, where T is a linear transformation of the fiber. The linear map is called a holonomy transformation.

Side Bar. For those familiar with covering spaces, this should look very familiar. This was the exact process of describing the action of the fundamental group of the base space on a covering space.

Consequences of Solutions to Ds=0

Suppose that we have k solutions to $Ds = 0$, say $s_1, \ldots, s_k$, such that over $U \subseteq B$, $\{s_i\}$ defines a local frame. Then given any $e \in E_b$, we may write $e = \sum \lambda_i s_i(b)$. Define $s_e = \sum \lambda_i s_i$. Consider the map $U \subseteq B \to E$ defined by s_e, and its induced map on tangent spaces, $s_{e*} : T_{b'}B \to T_{s_e(b')}E$.

Claim. *$s_e(U)$ is a submanifold of E passing through e. Furthermore $(s_e)_*(T_{b'}B = H_e \subset T_{s_e(b')}E$.*

We have a map $T_bB \to T_eE$ by $X \sim [\gamma(t)] \mapsto [\tilde{\gamma}_h(t)] = \tilde{X}_h$. We call $\tilde{X}_h$ the horizontal lift of X. The image is a dimension n subspace of T_eE, complementary to $V_e \subseteq E$; i.e, the image defines the horizontal subspace H_e. The second part of the claim is just that $s_e^*(T_{b'}B) = H_e$. The first part of the claim, is infact, true for any section. That is, if $s : B \to E$ is a section, then s locally looks like $x \mapsto (x, \sigma(x))$, e.g, locally s is of the form $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^k$. Consider the matrix of partial derivatives of s,

$$s_* = \begin{pmatrix} \mathrm{Id}_n \\ \frac{\partial \sigma_i}{\partial x_j} \end{pmatrix}$$

The rank of s_* will always be n and so the image of s is a dimension n submanifold. Take $X \sim [\gamma(t)]$. Then $s_e^*(X) \sim [s_e(\gamma(t))]$. So we just need to check that $s_e(\gamma(t))$ is a horizontal lift. But, this is true by the definition of s.

Hence, the image is contained in H_e and the map is injective via our local desciption. So, the dimensions also match up and we get that $s_e^*(T_{b'}B) = H_e$.

Summary: $H_e \subseteq T_eE$ defines a $n-$dimensional subspace. Taken together,

$$\bigcup_{e \in E} H_e$$

defines a *distribution*, $H \subseteq TE$. Given k linearly independent solutions to $Ds = 0$, we can conclude that H is an integrable distribution; that is, at each $e \in E$, there is a submanifold, $\mathcal{H}$ passing through e and such that $T\mathcal{H} = H_e$.

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu