THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 2

Professor Steven Bradlow
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University of Illinois at Urbana-Champaign

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In the previous lecture, we gave various bundles without a formal definition.

Definition 1. A bundle is a quadruple, (E, B, F, π), where E, B, F are spaces and $\pi : E \to B$ is a continuous map, called the projection, such that for every $x \in B$ we have that $\pi^{-1}(x) \cong F$ and for every $b \in B$ there is an open neighborhood $U \subseteq B$ of b such that $\pi^{-1}(U) \cong U \times F$ in a fiber preserving way. E is called the total space, B the base spaces, and F the fiber.

While the definition of a bundle is a very general one, we will be applying the definition of a bundle to several specialized categories.

(a) **Smooth**: E, B, F are smooth manifolds, maps are smooth.
(b) **TopM**: E, B, F are manifolds, maps are continous maps.
(c) **Holomorphic**: E, B, F are smooth complex manifolds, maps are holomorphic.

Suppose that (E, B, F, π) is a bundle. We identify two special cases by placing restrictions on F and π.

Definition 2. If F is a linear vector space (eg $\mathbb{R}^n, \mathbb{C}^n$) and the identifications $\pi^{-1}(U) \cong U \times F$ are linear maps, then we call (E, B, F, π) a vector bundle.

The tangent, normal, and tautological bundles are all vector bundles (*cf. Lecture 1*).

Definition 3. If F is a Lie group which has a smooth right action of E such that
(a) The action is free (i.e, $e \cdot g = e$ if and only if g is the identity element)
(b) The action preserves the fibers of $E \to B$.
We then call (E, B, F, π) a principal $F-$bundle.

The Hopf bundle is an example of a principal S^1 bundle and the homogeneous bundle $O(n) \to O(n)/O(n-1)$ is a principal $O(n-1)$ bundle.

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu

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