THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 20

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CLASS NOTES FROM MATH 433

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Curvatures

Recall from last time that the curvature of a connection D is $F_D \in \Omega^0(B, \Lambda^2 T^*M \otimes \mathrm{End}(E)) \equiv \Omega^2(B, \mathrm{End}(E))$. If $D = d + A$ with respect to a local frame, then the local formula for F is $F_A = dA + A \wedge A$.

Proposition. *Given a connection $D : \Omega^0(E) \to \Omega^1(E)$ on E, there exists a unique extension (called the* **covariant derivative***) of D to $D : \Omega^p(E) \to \Omega^{p+1}(E)$ such that (i) D is linear and (ii) For any $\alpha \in \Omega^q(B)$ and $\sigma \in \Omega^p(E)$, $D(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^q \alpha \wedge D\sigma$.*

Proof: Write $\sigma = \sum \beta_i \otimes s_i$, where $\beta_i \in \Omega^p(B)$ and $s_i \in \Omega^0(E)$. Then

$$\begin{aligned} D\sigma &= \sum D(\beta_i \otimes s_i) \\ &= \sum d\beta_i \otimes s_i + (-1)^p \beta_i \wedge Ds_i. \end{aligned}$$

□

Claim. *$F_D = D \circ D = D^2$, that is, $F_D(s) = D(Ds)$, for any $s \in \Omega^0(B, E)$. Here the second D is the extension of D to $\Omega^1(B, E) \to \Omega^2(B, E)$.*

Proof: First look at $\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E)$. With respect to local frame $\{e_i\}$ for E, say $D = d + A$ and $s = \sum_i s_i e_i$ is a section over U_α. Then $Ds = \sum (ds_i + A_{ij} s_j) e_i$. Therefore

$$\begin{aligned} D(Ds) &= -ds_i \wedge A_{ji} e_j + d(A_{ij} s_j) e_i - A_{ij} s_j \wedge A_{ki} e_k \\ &= -ds_i \wedge A_{ji} e_j + (dA_{ij}) s_j e_i - A_{ij} \wedge ds_j e_i + (A_{ki} \wedge A_{ij}) s_j e_k \\ &= \sum_{ij} (dA + A \wedge A)_{ij} s_j e_j \\ &= F_D(s). \end{aligned}$$

□

Question: What does F_D measure?

<u>First interpretation of curvatures</u>:

For the "complex"

$$(*) \qquad \Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E) \to \cdots,$$

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$D^2 \neq 0$, in fact, $D^2 = F_D$. So the curvature measures the failure of $(*)$ to be a complex. (Unlike in the de Rham complex $C^\infty \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots$, for which $d^2 = 0$!)

Note. If E admits D with $F_D = 0$ (such a D is called **flat**), then we can pick a local frame $\{e_i^\alpha\}$ such that $De_i^\alpha = 0$. Thus $A^\alpha = 0$, i.e. $D = d$ over U_α. Hence it follows that $g_{\alpha\beta}$ is constant. In other words "existence of a flat connection $\Leftrightarrow$ existence of a flat structure".

Remark. Even if $F_D = 0$, we can still have holonomy. However, if $F_D = 0$, then holonomy depends only on the homotopy type of loops. It follows that we can get a representation of $\pi_1(B)$ in $\mathrm{GL}(k,\mathbb{R})$, (called the **holonomy representation**) by $[\gamma] \mapsto$ holonomy around γ.

Note. The converse is also true, i.e. given a representation of $\pi_1(B)$ in $\mathrm{GL}(k,\mathbb{R})$, we get a flat $\mathrm{GL}(k,\mathbb{R})$ bundle.

Second interpretation of curvatures:

Proposition. *If X and Y are vector fields on B, then*

$$F_D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

In particular, if $[X,Y] = 0$, then $F_D(X,Y) = D_X D_Y - D_Y D_X$, that is, F_D measures the failure of D_X and D_Y to commute.

Proof: Fix $b \in B$. The curvature $F_D(X,Y)$ depends only on values at B, so we can use local description: $F_D(X,Y) = dA(X,Y) + A \wedge A(X,Y)$. For 1-forms α and β,

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{1}$$

and

$$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{2}$$

Then apply (1) and (2) to dA and $A \wedge A$. On the other hand,

$$\begin{aligned} D_X(D_Y s) &= D_X(d_Y s + A(Y)s) \\ &= d_X(D_Y + As) + A(X)(d_Y s + A(Y)s) \\ &= X(Y(s)) + X(A(Y))s + A(Y)X(s) + A(X)Y(s) + A(X)A(Y)s \\ &= \cdots \text{etc.} \end{aligned}$$

The rest of the proof is left as an exercise. □

Exercise 1. Finish the above proof.

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