THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 21

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Class Notes From Math 433

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March 11, 1998

If a vector bundle $E \to B$ admits a connection D such that $F_D = 0$, then we can pick a local frame $\{e_i^\alpha\}$ on U_α shch that $De_i^\alpha = 0$ $(A^\alpha = 0)$.

If $\gamma(t)$ is any curve in B (with $\gamma(0) = b$) we can define $\tilde{\gamma}_e(t)$, the horizontal lift through $\tilde{\gamma}_e(0) = e(\in E_b)$.

Note. With respect to a parallel frame $\{e_i^\alpha\}$, $\tilde{\gamma}_e(t) = \sum \tilde{\gamma}_i^\alpha(t) e_i^\alpha(\gamma(t))$ such that $\dot{\tilde{\gamma}}_i^\alpha(t) + A_{ij}^\alpha(\dot{\gamma}(t))\tilde{\gamma}_j^\alpha(t) = 0$. But $A_{ij}^\alpha = 0$, that is, $\dot{\tilde{\gamma}}_i^\alpha(t) = 0$, so $\tilde{\gamma}_i^\alpha(t)$ is a constant.

Now suppose $\gamma(t)$ is a loop, i.e. $\gamma(1) = \gamma(0) = b$. The holonomy around the loop $\gamma(t)$ is due to the fact that $\tilde{\gamma}_e(1)$ need **not** agree with $\tilde{\gamma}_e(0)$.

Question: How come $\tilde{\gamma}_e(1) \neq \tilde{\gamma}_e(0)$?

Answer: The path $\gamma(t)$ may not lie in one patch U_α! Suppose $\gamma(t)$ is covered by $U_0, U_1, \cdots, U_N$. Then we pick $0 < t_1 < t_2 < \cdots < t_{N+1} < 1$ such that $\gamma(t_\alpha) \in U_{\alpha-1} \cap U_\alpha$, for $\alpha = 1, \cdots N$, and $\gamma(t_{N+1}) \in U_N \cap U_0$. Say

$$\tilde{\gamma}_e(0) = \sum \tilde{\gamma}_i^0 e_i^0(0)$$
$$\tilde{\gamma}_e(t_1) = \sum \tilde{\gamma}_i^1 e_i^1(t_1)$$
$$\vdots$$
$$\tilde{\gamma}_e(t_\alpha) = \sum \tilde{\gamma}_i^\alpha e_i^\alpha(t_\alpha).$$

Then the horizontal lifting condition says $\tilde{\gamma}(t) = \sum \tilde{\gamma}_i^0 e_i^0(t)$, for $0 < t \leq t_1$. ($\tilde{\gamma}$ is constant w.r.t. $\{e_i^0(t)\}$.) At t_1, switch to frame $\{e_i^1(t)\}$: so $\tilde{\gamma}(t_1) = \sum (g_{ij}^{10}\tilde{\gamma}_j^0) e_j^1(t_1)$, that is, $(\tilde{\gamma}_i^1) = g_{ij}^{10}\tilde{\gamma}_j^0$. Therefore

$$\tilde{\gamma}^\alpha = g^{\alpha\alpha-1} \circ \tilde{\gamma}^{\alpha-1}.$$

At each overlap, pick up $g^{\alpha\alpha-1}$. Then, after going around the loop,

$$\tilde{\gamma}^0 \to \underbrace{\left[g^{0N} \circ g^{NN-1} \circ \cdots \circ g^{10}\right]}_{\text{holonomy!}} \circ \tilde{\gamma}^0.$$

Note. If the loop $\gamma(t)$ remains within a single patch, then $\tilde{\gamma}(1) = \tilde{\gamma}(0)$, that is, the holonomy is trivial—*no holonomy around small loops!*

Corollary. *Homotopy paths produces same holonomy.*

Thus we get a holonomy representation $\rho : \pi_1(B) \to \mathrm{GL}(k)$.

Conversely, given $\rho : \pi_1(B) \to \mathrm{GL}(k)$, we can construct a (flat) vector bundle $E \to B$ which admits D such that $F_D = 0$ and it produces ρ as holonomy representation as follows:

Construction: Let $\tilde{B} \xrightarrow{p} B$ be the universal cover and $\pi_1(B)$ acts on $\tilde{B}$ as covering transformation group. From $E = \tilde{B} \times_\rho \mathbb{R}^k \underset{\text{DEF}}{=} \tilde{B} \times \mathbb{R}^k/\pi_1(B)$ with $(\tilde{b}, v) \sim (\gamma\tilde{b}, \rho(\gamma)v)$.

Proposition. *(1) $E \to B$ is a flat $\mathbb{R}^k$-bundle.*

(2) $E \to B$ can be given a flat connection which induces $\rho : \pi_1(B) \to \mathrm{GL}(n)$ as holonomy representation.

Exercise 1. Prove the above proposition.

Note. Given $\rho : \pi_1(B) \to G$, then (1) $P = \tilde{B} \times_\rho G$ is a principal G-bundle. (2) P is flat. (3) P admits a flat connection.

Bianchi Identity

Given a connection D on a vector bundle $E \to B$ with curvature F_D, say locally $D = d + A$ and $F_D = F_A = dA + A \wedge A$ over U. Then F_A is a matrix of 2-form on U and we can compute

$$\begin{aligned} dF_A &= dA \wedge A - A \wedge A \\ &= (F_A - A \wedge A) \wedge A - A \wedge (F_A - A \wedge A) \\ &= F_A \wedge A - A \wedge F_A. \end{aligned}$$

That is, F_A satisfies the identity

$$dF_A + A \wedge F_A - F_A \wedge A = 0. \qquad \text{(Bianchi Identity)}$$

If we define $[A, F] = A \wedge F_A - F_A \wedge A$, then

$$dF_A + [A, F] = 0. \qquad \text{(Bianchi Identity)}$$

This can be given a global description: $D(F_D) = 0$, but we need to make sense of $D(F_D) = 0$. For this we need to look at *induced connections*!

Induced Connections I

(1)Induced Connections on $E_1 \oplus E_2$:

For connections D_i on E_i, $\forall i = 1, 2$, take $D^{\oplus} = D_1 \oplus D_2$. If locally $D_i = d + A_i$, then

$$D^{\oplus} = d + \begin{pmatrix} A_1 & * \\ * & A_2 \end{pmatrix}$$

with respect to direct sums of frames. That is, if $s = s_1 \oplus s_2 \in \Omega^0(E_1 \oplus E_2)$, then $D^{\oplus}(s_1 \oplus s_2) = D_1(s_1) \oplus D_2(s_2)$.

Exercise 2. Check that this is a connection and

$$F_A^{\oplus} = \begin{pmatrix} F_{A_1} & * \\ * & F_{A_2} \end{pmatrix}.$$

(2)Induced Connections on $E_1 \otimes E_2$:

For connections D_i on E_i, $\forall i = 1, 2$, set $D^{\otimes}(s_1 \otimes s_2) = D_1(s_1) \otimes s_2 + s_1 \otimes D_2(s_2)$. Define the connection

$$D^{\otimes} = D_1 \otimes I_2 + I_1 \otimes D_2.$$

What is the curvature $F_D^{\otimes}$?

Exercise 3. Show that $F_D^{\otimes} = F_1 \otimes I_2 + I_1 \otimes F_2$, that is, $F_D^{\otimes}(s_1 \otimes s_2) = F_1(s_1) \otimes s_2 + s_2 \otimes F_2(s_2)$.

(3)Induced Connections on E^*:

Given a connection D on $E \to B$, we can define a connection D^* on E^* such that if $s^* \in \Omega^0(E^*)$ and $t \in \Omega^0(E)$, then $[s^*(t)]_b = s_b^*(t_b), \forall b \in B$, is a smooth funtion on the base B. So we can take the differential

$$ds^*(t) = (D^* s^*)(t) + s^*(D(t)).$$

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