THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 23

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Given a connection on E, we got a connection on End(E)=Hom(E, E) $\cong E \otimes E^*$. With respect to a choice of a local frame, we saw that $D(h) = dh + [A, h]$.

Exercise 1. Show that if $h \in \Omega^0(\text{End}(E))$ and $s \in \Omega^0(E)$, then $D(hs) = D(h)s + hD(s)$. Here, $D(h) \in \Omega^1(\text{End}(E))$ and $D(s) \in \Omega^1(E)$. Along a vector field, $V \in \Omega^0(TB)$, show that

$$D_V(hs) = D_V(h)(s) + hD_V(s)$$

Last time we saw that given a vector bundle, $E \to B$, a map, $f : X \to B$, and a connection D on E, we could obtain a connection, f^*D on the pullback bundle $f^*(E)$: If $\{e_i\}$ is a local frame, we can define the pullback local frame by $\{f^*(e_i)\}$ in the natural way. With respect to the local frame $\{e_i\}$, we have that $D = d + A$. With respect to the pullback local frame, we gave $f^*(D) = d + A^*$ where A^* is the pull back of the connection one form, A.

Exercise 2. Show that if $s \in \Omega^0(E), V \in \Omega^0(TX)$, and $f_*(V) = W \in \Omega^0(TB)$, then

$$f^*(D)_V(f^*s) = f^*(D_W(s))$$

Show that $f^*(D)$ is the unique connectionon $f^*(E)$ with this property.

Consider the case when $X = (a, b)$ and $\gamma : (a, b) \to B$ is a curve. Then $\gamma^*(E)$ is the restriction of E to $\gamma(t)$. A section $\tilde{s} \in \Omega^0(\gamma^*(E))$ is a section of E along γ; i.e. a lift of γ to E. We have that $\gamma_*(\frac{\mathrm{d}}{\mathrm{dt}}) = \dot{\gamma}(t)$, the velocity vector field along $\gamma(t)$.

Definition 1. We make a special notational change for this case by

$$\gamma^*(D)_{\frac{\mathrm{d}}{\mathrm{dt}}}(\tilde{s}(t)) = \frac{\mathrm{D}}{\mathrm{dt}}\tilde{s}(t)$$

Then,

$$\frac{\mathrm{D}}{\mathrm{dt}}\tilde{s}(t) = \mathrm{D}_{\dot{\gamma}}\tilde{s}(t)$$

is the covariant derivative along the velocity vector field of γ.

Observe that $\tilde{s}(t)$ is a horizonal lift if and only if as a section of $\gamma^*(E)$, $\tilde{s}(t)$ is parallel with respect to $\gamma^*(D)$.

Note. In general, not every section of $f^*(E)$ is of the form $f^*(s)$ where s is some section of E. For example, if $x_1 \neq x_2$ are such that $f(x_1) = f(x_2)$ and s' is a section of $f^*(E)$ such that $s'(x_1) \neq s'(x_2)$, then s' will definitely not be the pullback of any section of E.

By contrast to the above remark, if $h : E \to E'$ is a bundle isomorphism of vector bundles over B, then we do get a (linear) bijection $\Omega^0(E) \to \Omega^0(E')$ in the natural way. So, we can define $h^*(D') = h^{-1} \circ D \circ h$. That is, $h^*(D')(h^{-1}(s)) = h^{-1}(D'(s))$.

Exercise 3. Suppose that $\{e_i\}$,$\{e_i'\}$ are local frames for E and E'. Then $h^*(D)(e_i) = h^{-1}D(h_{ji}e_j')$. Show that

$$h^*(D) = d + h^{-1}A'h + h^{-1}dh$$

Show that

$$F_{h^*(D)} = h^{-1} \circ F_{A'} \circ h$$

where $F_{A'}$ is the curvature 2 form for A'.

Exercise 4. IF we identify E and E', then we have an induced connection on $\mathrm{End}(E)$. Show that

$$h^*D = h^{-1}Dh = D + h^{-1}D(h)$$

where the second D is the induced connection.

Connections and Metrics

Suppose that the bundle, $E \to B$ is equipped with a metric, $\langle \ \ , \ \ \rangle$. We make the following definition regarding how the connections interact with the metric.

Definition 2. Call the connection D compatible with the metric $\langle \ \ , \ \ \rangle$ if for every pair $s, t \in \Omega^0(E)$ we have that

$$d\langle s, t\rangle = \langle Ds, t\rangle + \langle s, Dt\rangle$$

Note. For fixed $s, t \in \Omega^0(E)$, $\langle s, t\rangle$ defines a C^∞ function on B by $\langle s, t\rangle(b) = \langle s(b), t(b)\rangle$ and so we may certainly differentiate it. Furthermore, $\langle \ \ , \ \ \rangle$ is a bilinear pairing, $\Omega^0(E) \times \Omega^0(E) \to C^\infty(B)$. We can extend this to a map, $\Omega^1(E) \times \Omega^0(E) \to \Omega^1(B)$ by

$$\langle \alpha \otimes s, t\rangle \mapsto \alpha\langle s, t\rangle$$

In fact, we may fully extend to a map $\Omega^p(E) \otimes \Omega^q(E) \to \Omega^{p+q}(B)$ by

$$\langle \alpha \otimes s, \beta \otimes t\rangle \mapsto (\alpha \wedge \beta)\langle s, t\rangle$$

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