THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 24

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CLASS NOTES FROM MATH 433

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Recall that given a bundle, $E \to B$, a bundle metric, $\langle \ , \ \rangle$, and a connection D, we say that the metric is compatible with the connection if for all $s, t \in \Omega^0(E)$ the following product formula holds

$$d\langle s, t\rangle = \langle Ds, t\rangle + \langle s, Dt\rangle$$

Such a connection is sometimes called an *orthogonal* connection.

If E is a complex bundle and $\langle \ , \ \rangle$ is a Hermitian metric, then the connections which satisfy the above compatibility condition are called *unitary* connections.

Suppose that $\{e_i\}$ is a local frame for E with $\langle e_i, e_j\rangle = \delta_{ij}$ and that with respect to this choice of a local frame, $D = d + A$. Then, we have that

$$\begin{aligned} 0 = d\langle e_i, e_j\rangle &= \langle A_{ki} \otimes e_k, e_j\rangle + \langle e_i, A_{kj} \otimes e_k\rangle \\ &= A_{ki}\delta_{kj} + A_{kj}\delta_{ki} \end{aligned}$$

Hence, $A_{ji} + A_{ij} = 0$.

Note. In the complex case, using a Hermitian metric we get $A_{ji} + \bar{A}_{ij} = 0$.

This says that for an orthogonal connection, the connection 1-form, A, is skew symmetric with respect to an orthogonal frame. (If Lie(O(n)) denotes the Lie algebra on O(n), then we can identify the skew symmetric matrices with Lie(O(n))).

Similarly, for unitary connections on complex bundles, the connection 1-forms take their values in Lie(U(n)) when computed in a unitary frame.

Note. We can realize E as an associated principal O(k)-bundle. This turns out to be a general feature of connections on principal $G-$bundles: The connection 1-forms take their values in Lie(G).

A direct computation show that $F_A = dA + A \wedge A$ satisfies $F_A + F_A^t = 0$, again with respect to an orthogonal frame. Thus for any vector fields $X, Y \in TB$, the bundle map $F(X, Y) : E \to E$ is an isometry, i.e, with respect to orthogonal frames, $F(X, Y)$ is O(n)-valued.

An Alternative Description of the Compatibility Condition

A bundle metric, $\langle \ , \ \rangle$, can be viewed as a section of $E^* \otimes E^*$. On a vector space, $\langle \ , \ \rangle$ is a bilinear map, $V \times V \to \mathbb{R}$, which is equivalent to a map $V \to V^*$. Thus, on E, $\langle \ , \ \rangle$ is equivalent to a map, $E \to E^*$; that is, to a section of $\text{Hom}(E, E^*) \cong E^* \otimes E^*$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Given a local frame, $\{e_i\}$ of E, we define

$$H = \sum H_{ij} e_i^* \otimes e_j^*$$

where $H_{ij} = \langle e_i, e_j \rangle$. So, $H(s,t) = \langle s, t \rangle$. But, given a connection, D, on E, we get an induced connection on $E^* \otimes E^*$, denoted also by D. So, we can evaluate $D(H)$.

Exercise 1. Show that the following are equivalent:
(1) $d\langle s, t \rangle = \langle Ds, t \rangle + \langle s, Dt \rangle$.
(2) $D(H) = 0$.

Condition 2 above is sometimes called a covariant constant condition. That is, called H covariant constant if $D(H) = 0$.

Connections on TM - The Tangent Bundle

Recall that $TM \to M$ is a vector bundle with rank the dimension of M. We can define connections,

$$D : \Omega^0(TM) \to \Omega^0(T^*M \otimes TM) = \Omega^1(TM)$$

Sections of TM are just vector fields. In this case, $D_V(s)$ is the covariant derivative along the vector field V of the section s. But, V and s are now the same sort of creature: Vector Fields. When speaking of connections on a tangent bundle, we use the symbol ∇, rather than D. As before, $F_\nabla \in \Omega^2(\text{End}(TM))$ and given $X, Y \in \Omega^0(TM)$, we get $F_\nabla(X,Y) : TM \to TM$.

Recall that given a frame, $\{e_i\}$ and a connection $D = d + A$ with respect to this frame, A is a matrix of 1-forms. Write

$$A = \sum_{\alpha=1}^{n} A_\alpha dx_\alpha$$

with respect to the coordinates $(x_1, \ldots, x_n)$, where A_α is a matrix (map on the fibers of E). For $E = TM$, we can take

$$\{\frac{\partial}{\partial x_i}\}_{i=1}^m,$$

where m is the dimension of M, as a local frame for TM over the coordinate patch where the coordinates are $(x_1, \ldots, x_n)$. If $\nabla = d + A$, then

$$\nabla(\frac{\partial}{\partial x_i}) = A_{ji}(\frac{\partial}{\partial x_i}) = A_{ji,k} dx_k \otimes \frac{\partial}{\partial x_j}$$

This is usually written $\Gamma_{ji}^k dx_k \otimes \frac{\partial}{\partial x_j}$. The Γ_{ji}^k are called the *Cristofell* symbols. The curvature of ∇ is usually written, for maximal confusion, as Ω.

Parallel Transport and Horizontal Lifting

Given a curve, $\gamma(t)$, in M, a lift to TM is a vector field along $\gamma(t)$. Given such a lift, says $\tilde{\gamma}(t)$, we can compute the covariant derivative with respect to $\dot{\gamma}(t)$ along $\tilde{\gamma}(t)$, $\nabla_{\dot{\gamma}(t)} \tilde{\gamma}(t)$. But, for our lift, $\tilde{\gamma}(t)$, we can simply take $\dot{\gamma}(t)$.

Definition 1. Say that $\gamma(t)$ is *geodesic* if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$.

With respect to local coordinates, $(x_1, \ldots, x_m)$, a curve $\gamma(t)$ has component functions, $(x_1(t), \ldots, x_m(t))$. Since $\nabla = d + \Gamma^k_{ij} dx_k$, we have the following exercise.

Exercise 2. Show that the geodesic condition yields a system of ordinary differential equations,

$$\ddot{x}_k(t) + \Gamma^k_{ij}\gamma(t)\dot{x}_i(t)\dot{x}_j(t) = 0$$

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