THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 25

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During the last lecture we began to examine connections on tangent bundles, $TM \to M$. If M has dimension m, then this is a rank m bundle over M. Given a connection ∇ and local coordinates, $(x_1, \ldots, x_m)$ on M, we can produce a local frame, $\{\frac{\partial}{\partial x_i}\}$ for TM and $\{dx_i\}$ for T^*M. We saw that

$$\nabla(\frac{\partial}{\partial x_i}) = \Gamma^k_{ji} dx_k \otimes \frac{\partial}{\partial x_j}$$

This was how we defined the Christoffel symbols. The geodesic equation was derived from examining covariant constant vector fields. That is, given a curve $\gamma(t)$ in M, we called $\gamma(t)$ geodesic if the velocity vector field, $\dot{\gamma}(t)$ was covariant constant, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. In local coordinates, $\gamma(t) = (x_1(t), \ldots, x_m(t))$, we get the following set of ordinary differential equations

$$\ddot{x}_k(t) + \Gamma^k_{ji}(\gamma(t))\dot{x}_i(t)\dot{x}_j(t) = 0, \qquad k = 1, \ldots, m$$

A consequence of the existence and uniqueness of solutions to differential equations implies that given any vector $\vec{v} \in T_x M$, there exists a unique geodesic curve, $\gamma_{\vec{v}}(t)$ with $\gamma_{\vec{v}}(0) = x$ and $\dot{\gamma}_{\vec{v}}(0) = \vec{v}$.

Exercise 1. Let $\gamma_{\vec{v}}(t)$ be the geodesic through x in the direction of $\vec{v}$. If λ is some constant, show that $\gamma_{\lambda\vec{v}}(t) = \gamma_{\vec{v}}(\lambda t)$.

Corollary. *Given a unit vector $\vec{u} \in T_x M$ (a unit vector with respect to some Riemannian metric), for small enough λ, $\gamma_{\lambda\vec{u}}(t)$ will be defined at $t = 1$. We define a map, $T_x M \to M$ defined on a small neighborhood of $0 \in T_x M$ by $\vec{v} \mapsto \gamma_{\vec{v}}(1)$. This map is called the exponential map and is denoted by $exp(\vec{v})$.*

A fact from Riemannian geometry says that this is a diffeomorphism of the neighborhood of 0 in $T_x M$ onto a neighborhood of x in M. The proof relies on the implicit function theorem. In order to set things up properly, we must examine the derivative of this map, $D_0(\exp) : T_0(T_x M) \to T_x M$.

Exercise 2. Show that $D_0(\exp) = \mathrm{Id}$.

This defined geodesic coordinates.

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Claim. *With respect to these geodesic coordinates, Γ_{ij}^k vanishes at x.*

Sketch of Proof: Given any $\vec{v} \in T_xM$, evaluate $(\Gamma_{ij}^k dx_k)(\vec{v})$ using the geodesic along $\vec{v}$, $\gamma_{\vec{v}}(t)$. Use the geodesic equation and note that in geodesic coordinates, $\gamma_{\vec{v}}(t) = t\vec{v} = (x_1(t), \ldots, x_m(t))$.

Torsion

Definition 1. For a connection, ∇, on the tangent bundle, TM, over M, the *torsion* of the connection is a tensor field in

$$\Omega^0(\wedge^2(T^*M) \otimes TM)$$

defined by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for any vector fields, $X, Y \in TM$.

It is not at all clear that τ is actually an element of $\Omega^0(\wedge^2(T^*M) \otimes TM)$. The following exercise is in this direction.

Exercise 3. Show that $\tau(X,Y)_b$ depends only on the values of X_b and Y_b. This will show that $\tau_b \in \wedge^2 T_b^* M \otimes T_bM$. Show that if f is a function, then $\tau(fX, Y) = f\tau(X,Y)$.

What does the torsion measure? With respect to the frames $\{\frac{\partial}{\partial x_i}\}$ for TM and $\{dx_i\}$ for T^*M, we see that

$$\begin{aligned}\tau(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= \nabla_{\frac{\partial}{\partial x_i}}(\frac{\partial}{\partial x_j}) - \nabla_{\frac{\partial}{\partial x_j}}(\frac{\partial}{\partial x_i}) \\ &= (\Gamma_{ji}^k - \Gamma_{ij}^k)\frac{\partial}{\partial x_k}\end{aligned}$$

So, if

$$\tau = \tau_{ij}^k dx_i \wedge dx_j \otimes \frac{\partial}{\partial x_k}$$

then $\tau_{ij} = \Gamma_{ij}^k - \Gamma_{ji}^k$. If $\tau = 0$, then the connection has symmetric Christoffel symbols.

Fact: Given ∇, if $\tau \neq 0$, then we can modify ∇ to obtain a new connection, $\tilde{\nabla}$, with $\tilde{\tau} = 0$. The modification is given by the following procedure. We have

$$\Omega^1(\mathrm{End}(TM)) = \Omega^0(T^*M \otimes \mathrm{End}(TM))$$

and

$$T^*M \otimes \mathrm{End}(TM) \cong T^*M \otimes (TM^* \otimes TM) \cong (T^*M \otimes TM^*) \otimes TM$$

Since $\wedge^2(T^*M) \subseteq T^*M \otimes TM^*$, we have that $\tau \in \Omega^1(\mathrm{End}(TM)$. We can use this to define $\tilde{\nabla} = \nabla - \frac{1}{2}\tau$.

Exercise 4. Show that the torsion of $\tilde{\nabla}$ is zero.

The Levi-Civita Connection

Recall that a metric, g, on a manifold M if it is a bundle metric on the tangent bundle of M. We can thus ask for connections on $TM \to M$ to be compatible with the metric. By our discussion of orthogonal connections, this can be expressed by the condition that $\nabla g = 0$.

Claim. *Given g, there is a unique connection, called the Levi-Civita connection and denoted ∇^{lc}, such that $\nabla^{lc} g = 0$ and ∇^{lc} is torsion free.*

Proof: Use local coordinates $\{x_1, \ldots, x_n\}$. Write

$$g = \sum g_{ij} dx_i \otimes dx_j$$

with respect to the local frame $\{dx_i\}$ for T^*M. Set

$$\Gamma^k_{ij} = \frac{1}{2} g^{-1}_{kl} [g_{jl,i} - g_{ij,l} + g_{li,j}]$$

where $g_{jl,i} = \frac{\partial}{\partial x_i}(g_{jl})$ to form the desired connection. □

We now get induced connections on $T^*M, \otimes T^*M, \otimes TM$,etc. On $\wedge^p T^*M$, we have $\nabla^{lc} : \Omega^p(M) \to \Omega^0(T^*M \otimes \wedge^0 T^*M)$ and $d : \Omega^p(M) \to \Omega^{p+1}(M)$. We would like to relate these two maps.

Claim. *Alt*$(\nabla^{lc}) = d$ *where Alt is the alternation.*

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