THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 26

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Class Notes From Math 433

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March 30, 1998

Characteristic Classes I

Recall: Given a vector bundle $E \to B$, we have a connection

$$D : \Omega^0(E) \to \Omega^1(E)$$

with $D(fs) = df \otimes s + fDs$.

With respect to a local frame $\{e_i\}$: $s = \sum s_i(x)e_i(x) \sim (s_1, \cdots, s_n)^t$ and $D = d + A$. Thus

$$D(s) = d\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} + A\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}.$$

The connection extends to define a covariant derivative $\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E) \xrightarrow{D} \cdots$. The curvature of D is $F = D^2$, given by $F_A = dA + A \wedge A$ with respect to a local frame.

Our next topic is the use of curvature of a connection to define **Characteristic Classes** (devices for measuring the topology of bundles). Characteristic classes for a vector bundle $E \to B$ are:

(1) Classes in $H^*(B)$.
 (We will look only at classes in $H^*(B, \mathbb{R})$ or $H^*(B, \mathbb{C})$ since the classes we obtained will be described by differential forms.)
(2) Characteristic in the sense that "isomorphic bundles have same characteristic classes".
 More generally, classes behave naturally under bundle maps: Given

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

 so $E \cong f^*(E')$, we have $f^*(\text{Classes of } E' \text{ in } H^*(B')) = (\text{Classes of } E \text{ in } H^*(B))$.

Two approaches to construct characteristic classes:

(1) (Abstract) Using unversal bundles/classfying spaces, we can write

$$\begin{array}{ccc} f^*(EG) = E & & EG \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow[f]{} & BG \end{array}$$

we can define characteristic classes for $EG \to BG$ and then take f^*(universal characteristic classes).

(2) (More Concrete) Define using curvature of a connection. Locally we know that F_A is a 2-form of matrices. Thus given a polynomial function on matrices, say P, we can evaluate $P(F_A)$ and get a differential form. If this form is closed, we get a class $[P(F_A)]$ in the cohomology of B. If, moreover, this class is independent of the choices made (i.e. connection and local frame), then the class is characteristic of the bundle.

We will take <u>the second approach</u>.

There are three things need to do:

(a) Understand which P are suitable.

(b) Show that the forms $P(F_A)$ are closed.

(c) Show $[P(F_A)] \in H^*(B, \mathbb{R})$ is independent of D and of the local frames.

(a) Invariance with respect to frames.

Recall: Say A^α is a connection 1-form and F^α is a connection 2-form with respect to $\{e_i^\alpha\}$ and A^β is a connection 1-form and F^β is a connection 2-form with respect to $\{e_i^\beta\}$. Say $g^{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the transition function relating the local frames. Here G is the structure group of the bundle (e.g. $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{O}(n)$ for real bundles, $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{U}(n)$ for complex bundles). Thus $e_i^\alpha = [g^{\alpha\beta}]_{ji} e_j^\beta$. Then

$$F^\alpha = g^{\alpha\beta} \circ F^\beta \circ g^{\beta\alpha}.$$

Suppose that we have a polynomial

$$P : \mathrm{Mat}_n \to \mathbb{R} \text{ (or } \mathbb{C}).$$

For $P(F^\alpha)$ to be independent of local frames, we need $P(A^{-1}BA) = P(B)$, for all $A \in G$.

Examples of polynomial functions on matrices.

$$\begin{aligned} \mathrm{Tr}(A) &= \sum A_{ii} && \text{deg } 1 \\ \mathrm{Tr}(A^2) &= \sum_{i,j} A_{ij} A_{ji} && \text{deg } 2 \\ &\vdots \\ \det(A) &= \sum_\sigma (-1)^{|\sigma|} A_{1\sigma(1)} \cdots A_{n\sigma(n)} && \text{deg } n \end{aligned}$$

Exercise 1. Check that for all of these $P(A^{-1}BA) = P(B)$ for all $A \in \mathrm{GL}(n)$.

Note. We can also define $P(A)$ for form-valued A, using wedge products in the definition.

Definition 1. If $P(A^{-1}BA) = P(B)$ for all $A \in \mathrm{GL}(n, \mathbb{C})$, we say P is a $\mathrm{GL}(n, \mathbb{C})$-*invariant polynomial*, for $P : \mathrm{Mat}_n \to \mathbb{C}$.

Note. Classification of all such polynomials is an important part of characteristic classes theory.

Conclusion. If P is a $\mathrm{GL}(n)$-invariant polynomial and F^α is a local description of curvature, then $P(F^\alpha)$ is well defined global form on B (we will sometimes write this as $P(D)$). For P of degree m, $P(F^\alpha) \in \Omega^{2m}(B)$.

Key Result. *Let P be a $\mathrm{GL}(n)$-invariant polynomial and D any connection on a vector bundle $E \to B$. Then*

(i) $dP(D) = 0$.
(ii) $P(D_1) - P(D_2) = d(TP(D_1, D_2))$, for some form $TP(D_1, D_2)$.

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