THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 27

Professor Steven Bradlow
Class Notes From Math 433

University of Illinois at Urbana-Champaign

April 1, 1998

Characteristic Classes II

Recall: A polynomial $P : \mathrm{Mat}_n \to F$ ($F = \mathbb{R}$ *or* $\mathbb{C}$) is $\mathrm{GL}(n)$**-invariant** if $P(ABA^{-1}) = P(B)$, $\forall B \in \mathrm{Mat}_n$ and $A \in \mathrm{GL}(n)$.

We can apply P to form-valued matrices. In particular, if D is a connection on E and F^α is a local curvature 2-form, then $P(F^\alpha)$ is globally well defined (denote by $P(D)$).

Proposition. *For P an invariant polynomial and D any connection on a vector bundle $E \to B$, then*

(1) $dP(D) = 0$.

(2) If D_0 and D_1 are two connections, then $[P(D_0)] = [P(D_1)]$ in $H^*(B)$, that is, there exists a form $TP(D_0, D_1)$ such that $P(D_0) - P(D_1) = d(TP(D_0, D_1))$.

Preliminary: Assume $P(A)$ is homogeneous, say degree m.

Claim. *We can write $P(A) = P(A_1, \cdots, A_m)$, where $P(A_1, \cdots, A_m) : \mathrm{Mat}_n \times \cdots \times \mathrm{Mat}_n \to F$ is a multilinear and symmetric map, called the* **polarization of** *P.*

For example, if $P(A) = \mathrm{Tr}(A^2)$, then $P(A_1, A_2) = (\frac{1}{2})\,\mathrm{Tr}(A_1A_2 + A_2A_1)$

Recipe: If $\deg P$ is m, look at $P(\sum_{i=1}^m t_iA_i)$ and expand in power of t_i, then the polarization is given by

$$P(A_1, \cdots, A_m) = (\text{Coefficent of the term in } t_1 \cdots t_m) \cdot (\frac{1}{m!}) \ .$$

For example, $\mathrm{Tr}(t_1A_1 + t_2A_2)^2 = t_1^2\,\mathrm{Tr}(A_1^2) + t_1t_2\,\mathrm{Tr}(A_1A_2 + A_2A_1) + t_2^2\,\mathrm{Tr}(A_2^2)$, Thus $P(A_1, A_2) = (\frac{1}{2!})\,\mathrm{Tr}(A_1A_2 + A_2A_1)$.

Proof of claim: (Hint) Write

$$P(A) = \sum_{\substack{\#I = \#J = m \\ I = \{i_1, \cdots, i_m\}}} \lambda_{IJ} A_{i_1j_1} \cdots A_{i_mj_m} \ ,$$

Then look at $P(\sum_{i=1}^m t_iA_i)$. □

Thus

$$\begin{aligned} dP(F) &= d(P(F, \cdots, F)) \\ (Ex.!) &= P(dF, F, \cdots, F) + P(F, dF, F, \cdots, F) + \cdots + P(F, \cdots, F, dF) \\ &= mP(dF, F, \cdots, F). \end{aligned}$$

But by *Bianchi*, $dF + [A, F] = 0$, so $dF = -[A, F] = [F, A]$. So

$$dP(F) = mP([A, F], F, \cdots, F).$$

To evaluate this $dP(D)$ at $x_0 \in B$, we can pick a convenient frame.

Claim. *We can pick a frame such that $A(x_0) = 0$.*

Then $dP(F)(x_0) = mP(0, F, \cdots, F) = 0$.

Proof of claim: Suppose $D = d + A$ with respect to some local frame $\{e_i\}$ defined on U, and $A(x_0) \neq 0$. Let's define a new frame $e'_i = h_{ji} e_j$, where the matrix $[h_{ij}] : U \to \mathrm{GL}(n)$. Now with respect to $\{e'_i\}$, $A' = h^{-1} \circ A \circ h + h^{-1} dh$. Suppose we can find h_{ij} such that

$$\left.\begin{aligned} h(x_0) &= I \\ dh(x_0) &= -A(x_0) \end{aligned}\right\} .$$

Then $A'(x_0) = A(x_0) - A(x_0) = 0$!

Thus we must show that we can find $h : U \to \mathrm{GL}(n)$ such that

$$\left\{\begin{aligned} h(x_0) &= I \\ dh(x_0) &= -A(x_0) \end{aligned}\right. .$$

Think of this in the local coordinates $(y_1, \cdots, y_n)$ with $x = (0, \cdots, 0)$, define

$$h(y) = I - y_i A_i,$$

where $A = \sum A_i dy_i$. Then for y in suitably small neighborhood of 0, $\det h(y) \neq 0$, i.e., $h(y) \in \mathrm{GL}(n)$. □

Conclusion:

$$dP(D) = 0.$$

Suppose D_0 and D_1 are connections. Define

$$D_t = tD_1 + (1-t)D_0 \qquad t \in [0, 1].$$

Claim. *D_t is a connection, for all $t \in [0, 1]$.*

In fact, if $D_1 = D_0 + \theta$, with $\theta \in \Omega^1(\mathrm{End}(E))$, then

$$\begin{aligned} D_t &= D_0 + (t\theta) \\ &= D_0 + \theta_t. \end{aligned}$$

Let θ^α and F_t^α be local expressions for θ and F_t with respect to a local frame. So θ^α = matrix of 1-forms and F_t^α = matrix of 2-forms.

Note. $P(\theta^\alpha, F_t^\alpha, \cdots, F_t^\alpha)$ is well defined for any local expression and if $\deg P = m$, then this is a $(2m-1)$-form.

Proposition.

$$TP(D_0, D_1) = m \int_0^1 P(\theta, F_t, \cdots, F_t)\, dt \ .$$

Proof: By the Fundamental Theorem of Calculus,

$$P(D_1) - P(D_0) = \int_0^1 (\frac{d}{dt} P(F_t, \cdots, F_t))\, dt \ ,$$

and

$$\frac{d}{dt}P(F_t,\cdots,F_t) = mP(\dot{F}_t,F_t,\cdots,F_t)\ .$$

If $D_t = D_0 + t\theta$, then the curvature

$$\begin{aligned} F_t &= D_t \circ D_t \\ &= D_0^2 + t(D_0\theta + \theta D_0) + t^2\theta\wedge\theta. \end{aligned}$$

That is,

$$F_t(s) = F_0(s) + t(D_0(\theta(s))) + \theta(D_0(s)) + t^2\theta\wedge(\theta(s)) \qquad \text{for any section } s.$$

□

Exercise 1. Show that

$$F_t = D_0 + tD_0(\theta) + t^2\theta\wedge\theta.$$

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu