THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 28

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CLASS NOTES FROM MATH 433

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Recall that given a vector bundle $E \to B$, we define characteristic classes in $H^*(B;\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) by starting with a GL_n invariant polynomial, $P : \mathrm{Mat}_n \to \mathbb{F}$, and a connection, $D : \Omega^0(E) \to \Omega^1(E)$, and then defining $P(D) = P(F)$ where F is the curvature of the connection. We evaluate $P(F)$ on the local expression for the curvature of D. Last time we proved that $dP(F) = 0$ and so $[P(D)]$ represents a cohomology class in $H^*(B;\mathbb{F})$. Infact, if the degree of P is m, then $[P(D)] \in H^{2m}(B;\mathbb{F})$. What remains to be shown is that $[P(D)]$ is independent of the connection D.

Suppose that D_0, D_1 are connections. Define the path in the space of connections from D_0 to D_1 by $D_t = tD_1 + (1-t)D_0$. Then,

$$P(D_1) - P(D_0) = \int_0^1 \frac{d}{dt} P(F_t, \dots, F_t)\ \mathrm{dt}$$

where $P(A_1, \dots, A_m)$ is the polarization of P. [E.g, for $P(A) = \mathrm{Tr} A^2$, $P(A_1, A_2) = \frac{1}{2}\mathrm{Tr}(A_1A_2 + A_2A_1)$].

Claim.

$$\int_0^1 \frac{d}{dt} P(F_t, \dots, F_t)\ dt = d(TP(D_0, D_1))$$

where

$$TP(D_0, D_1) = m \int_0^1 P(\theta, F_t, \dots, F_t)\ dt$$

and $\theta = D_1 - D_0$*.*

Note. $\theta \in \Omega^1(\mathrm{End}(E))$. Thus, we can evaluate $P(\theta, F_t, \dots, F_t)$ by fixing a local frame and using a local expression for θ, F_t. By the invariance of P and the transformation properties of a section of $\mathrm{End}(E)$, $P(\theta, F_t, \dots, F_t)$ is independent of the local frame.

Suppose that with respect to a local frame

$$\begin{aligned} D_0 &= d + A_0 \\ D_1 &= d + A_1 \\ \theta &= A_1 - A_0 \end{aligned}$$

Then $D_t = d + A_0 + t\theta = D + A_t$ where

$$A_t = A_0 + t\theta \quad (1)$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Thus,

$$F_t = dA_t + A_t \wedge A_t \quad (2)$$

and

$$\dot{F}_t = d\dot{A}_t + \dot{A}_t \wedge A_t + A_t \wedge \dot{A}_t \quad (3)$$

We need

$$\frac{d}{dt} P(F_t, \ldots, F_t) = mP(\dot{F}_t, F_t, \ldots, F_t)$$

Exercise 1. IF $P : \mathrm{Mat}_n \times \ldots \mathrm{Mat}_n \to \mathbb{C}$ is multilinear, then

$$\frac{d}{dt} P(A_1(t), \ldots, A_m(t)) = \sum_{i=1}^{m} P(A_1(t), \ldots, \dot{A}_i(t), \ldots, A_m(t))$$

To evaluate $P(\dot{F}_t, F_t, \ldots, F_t)$ at (t_0, x_0), we pick a local frame such that $A_{t_0}(x_0) = 0$.(Recall by the Bianchi identity that we also get $(dF_{t_0})(x_0) = 0$.) Thus, at x_0 we get

$$\begin{aligned} \dot{F}_{t_0} &= d\dot{A}_{t_0} \qquad \text{by (3)} \\ &= d\theta \qquad \text{by (1)} \end{aligned}$$

Hence,

$$\int_0^1 \frac{d}{dt} P(F_t, \ldots, F_t) \text{ dt} = m \int_0^1 P(d\theta, F_t, \ldots, F_t) \text{ dt}$$

Also, (remembering that P is a multilinear map),

$$d(P(\theta, F_{t_0}, \ldots, F_{t_0})) = P(d\theta, F_{t_0}, \ldots, F_{t_0}) + (m-1)P(\theta, dF_{t_0}, \ldots, F_{t_0})$$

Evaluating this in the special frame described above, we see that

$$d(P(\theta, F_{t_0}, \ldots, F_{t_0})) = P(d\theta, F_{t_0}, \ldots, F_{t_0})$$

Hence,

$$\int_0^1 \frac{d}{dt} P(F_t, \ldots, F_t) \text{ dt} = d(m \int_0^1 P(\theta, F_t, \ldots, F_t) \text{ dt})$$

Example. Let $P_1(A) = \mathrm{Tr}(A)$. Then

$$P(D_1) - P(D_0) = d \int_0^1 (\mathrm{Tr}(\theta) \text{ dt} = d\mathrm{Tr}(\theta)$$

Suppose $P_2(A) = \mathrm{Tr}(A^2)$. Then

$$TP_2(D_0, D_1) = d \int_0^1 \mathrm{Tr}(\theta F_t + F_t \theta) \text{ dt} = 2d \int_0^1 \mathrm{Tr}(\theta F_t) \text{ dt}$$

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