THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 29

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Recall that given a vector bundle, $E \to B$ and any GL_n invariant polynomial P, we can define cohomology classes, $[P(E)] \in H^*(B; \mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. If I_{GL_n} is the set of all GL_n invariant polynomials on Mat_n, then we can think of I_{GL_n} as a ring under functional addition and multiplication and hence get a map, $I_{\mathrm{GL}_n} \to H^*(B; \mathbb{F})$ of rings, $P \mapsto [P(E)]$. This is called the *Chern-Weil* homomorphism.

Note. If E admits a flat connection, i.e, D such that $F_D =$, then $[P(E)] = 0$ for all P. Since every trivial bundle has $[P(E)] = 0$ and there are flat bundles which are not trivial, there exist bundles with the same characteristic classes but which are not isomorphic.

If $E' \to B'$ is a bundle and $f : B \to B'$ is a map with pullback bundle $f^*(E') = E \to B$, i.e, we have the following set up,

$$\begin{array}{ccc} E & & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' \end{array}$$

For any GL_n invariant polynomial P, we have $[P(E)] \in H^*(B; \mathbb{F}), [P(E')] \in H^*(B'; \mathbb{F})$, and we would like for these classes to match up; that is, for $[P(E)] = f^*[P(E')]$. Given a connection D' on E' we can define the pullback connection on E, $D = f^*(D')$. Since $F = f^*(F')$, we do indeed get $[P(E)] = f^*[P(E')]$ by definition.

The Complex Bundle Case

Question: What are all of the $\mathrm{GL}(n, \mathbb{C})$ invariant polynomials? That is, can we classify all of the polynomials, $P : \mathrm{Mat}(n, \mathbb{C}) \to \mathbb{C}$ which are invariant under conjugation by elements of $\mathrm{GL}(n, \mathbb{C})$?

Consider the case of

$$\begin{aligned} P \in \mathrm{DIAG} &= \{A \in \mathrm{Mat}(n, \mathbb{C}) : A \text{ is diagonalizable}\} \\ &= \{gAg^{-1} = \begin{pmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_n \end{pmatrix} : g \in \mathrm{GL}(n, \mathbb{C})\} \end{aligned}$$

If P is a $\mathrm{GL}(n,\mathbb{C})$ invariant polynomial, then by definition,

$$P(A) = P(\begin{pmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_n \end{pmatrix})$$

where the λ_i are the Eigenvaluues. Thus, P depends only on the λ_i.

Claim. *$P(\lambda_1,\dots,\lambda_n)$ must actually be a symmetric function of λ_i.*

Proof: By appropriate conjugation in $\mathrm{GL}(n,\mathbb{C})$, we can permute the λ_i. Since P is invariant under such conjugation, P must be invariant under the permutation of the λ_i. That is, P must be a symmetric function of the λ_i. $\square$

Example.

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Fact: All such P are generated by the elementary symmetric functions,

$$\begin{aligned} \sigma_0(\lambda_1,\dots,\lambda_n) &= 1 \\ \sigma_1(\lambda_1,\dots,\lambda_n) &= \sum_{i=1}^{n} \lambda_i \\ \sigma_2(\lambda_1,\dots,\lambda_n) &= \sum_{i,j} \lambda_i\lambda_j \quad i \neq j \\ &\dots \\ \sigma_n(\lambda_1,\dots,\lambda_n) &= \lambda_1\lambda_2\dots\lambda_n \end{aligned}$$

Example.

$$\sum_{i=1}^{n} \lambda_i^2 = (\sum_{i=1}^{n} \lambda_i)^2 - 2(\sum_{i\neq j} \lambda_i) = \sigma_1^2 - 2\sigma_2$$

This is a basic fact from algebra and essentially says that any symmetric polynomial can be written as

$$Q(\sigma_0(\lambda_1,\dots,\lambda_n),\dots,\sigma_n(\lambda_1,\dots,\lambda_n))$$

where Q is a polynomial.

Note.

$$P(A) = \det(I + A) = \det(\begin{pmatrix} 1+\lambda_1 & 0 & \dots & \dots & 0 \\ 0 & 1+\lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1+\lambda_n \end{pmatrix})$$

$$= \prod_{i=1}^{n}(1+\lambda_i) = 1 + \sigma_1 + \dots + \sigma_n$$

So, the σ_i arrise as the homogeneous parts of $P(A)$.

Note. So far, all of this applies equally to the real case.

We want not quite $\sigma_j(F)$, but rather $\sigma_j(\frac{i}{2\pi}F)$.

Definition 1. The *Chern classes* of E are defined by $c_k(E) = [\sigma_k(\frac{i}{2\pi}F)]$. c_k is called the *k-th Chern class of E*. These are the homogeneous parts of the *total Chern class*, $c(E) = [\det(I + \frac{i}{2\pi}F)]$.

We have that

$$\det(I + \frac{i}{2\pi}) = 1 + \mathrm{Tr}(\frac{i}{2\pi}F) + \cdots + \det(\frac{i}{2\pi}F)$$

Note. $c_k(E) \in H^{2k}(B; \mathbb{C})$. We will see, in fact, that $c_k(E) \in H^{2k}(B; \mathbb{R})$.

We still need to answer the question as to why it is enough only to look at diagonalizable matrices, DIAG.

Note. This is the point where the real case differs from the complex case.

(1) DIAG $\subset \mathrm{Mat}_n$ is dense. If $A \in \mathrm{Mat}_n$, then there is a sequence of A_i with each $A_i \in$ DIAG with

$$A = \lim_{i \to \infty} A_i$$

Since the invariant polynomials are continuous functions, we can evaluate

$$P(A) = \lim_{i \to \infty} P(A_i)$$

(2) Any complex bundle, $E \to B$, can be given a Hermitian metric. So, we can take a compatible connection, D and with respect to a unitary frame, F is skew Hermitian; i.e, $F + F^* = 0$. In general, $AA^* = A^*A$ implies that A is diagonalizable. So, if $F + F^* = 0$, then F is diagonalizable. So, for purposes of evaluating c_k, we can restrict to diagonalizable matrices.

Note. If $(\frac{i}{2\pi}F)^* = \frac{i}{2\pi}F$, then $(I + \frac{i}{2\pi}F)^* = I + \frac{i}{2\pi}F$. So,

$$\overline{\det(I + \frac{i}{2\pi}F)} = \det(I + \frac{i}{2\pi}F)^* = \det(I + \frac{i}{2\pi}F)$$

This says that $c(E) = \overline{c(E)}$ and hence that $c_n(E) \in H^{2k}(B; \mathbb{R})$ for all k.

Remark. The factors $\frac{1}{2\pi}$ are inserted so that the classes $c_k(E)$ are integer valued; i.e, $c_k(E) \in H^{2k}(B; \mathbb{Z})$.

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