THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 30

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Class Notes From Math 433

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April 13, 1998

Characteristic Classes for Complex Bundles II

Definition 1. Let $E \to B$ be a complex bundle of rank n. To define the *Chern classes* of E we use the $(\mathrm{GL}(n, \mathbb{C})$-invariant) polynomials corresponding to the homogeneous parts of $\det(I + A)$, i.e. we use $c_j(A)$ where

$$\det(I + A) = \sum_{j=0}^{n} c_j(A).$$

We define the *j-th Chern class* of E by

$$c_j(E) = c_j(\frac{i}{2\pi}F) \ \in H^{2j}(B, \mathbb{C}),$$

where F is the curvature form of any connection on E, expressed with respect to any local frame. We define the *total Chern class* of E by

$$c(E) = \sum_{j=0}^{n} c_j(E) = \det(I + \frac{i}{2\pi}F).$$

Note. We can always

- fix Hermitian metric on E
- take D unitary
- use unitary frames

Then $F + F^* = 0$. (Thus if $F + F_{ij}\, dx_i \wedge dx_j$ with respect to local coordinates, then $F_{ij} + F_{ij}^* = 0$.)

Recall: Any $A \in \mathrm{U}(n) = \mathrm{Lie}(\mathrm{U}(n))$ is diagonalizable by $T \in \mathrm{U}(n)$, so we can restrict our attentation to

$$\boxed{\mathrm{U}(n)\text{-invariant polynomials on } \mathrm{U}(n)}$$

Claim. *The Chern classes are real cohomology classes, i.e.* $c_j(E) \in H^{2j}(B, \mathbb{R})$.

That is, given any $X_1, \cdots, X_{2j} \in \Omega^0(TB)$, $c_j(E)(X_1, \cdots, X_{2j}) \in \mathbb{R}$. (In fact, the classes are **integral**, i.e. $c_j(E)[\sigma] \in \mathbb{Z}$ for $[\sigma] \in H_{2j}(B, \mathbb{Z})$, but we won't prove that here.)

Proof: If $F + F^* = 0$, then $\frac{i}{2\pi}F = (\frac{i}{2\pi}F)^*$. Thus $\det(I + \frac{i}{2\pi}F) = \overline{\det(I + \frac{i}{2\pi}F)}$. Therefore $c_j(E) = \overline{c_j(E)}$, for all j. (Explicitiy: if $c_j(E) = \left[\sum \gamma_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_{2j}}\right]$, then $\gamma_I = \overline{\gamma_I}$.) $\square$

Example 1. For a line bundle $\gamma \to \mathbb{P}^1$ (recall $\gamma|_{[z_0,z_1]}$ = line through (z_0, z_1)), $c_0(\gamma) = \det(I) = 1$ and $c_1(\gamma) = \mathrm{Tr}(\frac{i}{2\pi}F) = \frac{i}{2\pi}F \in H^2(\mathbb{P}^1, \mathbb{R})$. (In general, $c_j(E)$ is defined for $j \leq \min(\frac{\dim B}{2}, \mathrm{Rank}(E))$.) Then

$$\int_{\mathbb{P}^1} c_1(\gamma) = c_1(\gamma)\,[\mathbb{P}^1] = -1$$

(we will see the proof later!)

Before we get to more computations, we look at general properties:

Formulae: From definition and expression for $\det(I + \frac{i}{2\pi}A) = \sum c_j(A)$, we can see

$$\begin{aligned} c_0(A) &= \det I = 1 \\ c_1(A) &= \mathrm{Tr}(\frac{i}{2\pi}A) \\ &\vdots \\ c_n(A) &= \det(A) \end{aligned}$$

[if we do the computation for $I + \frac{i}{2\pi}A$ diagonal, and we write the diagonal matrix for $I + \frac{i}{2\pi}A$ as

$$\begin{pmatrix} 1+x_1 & & \\ & \ddots & \\ & & 1+x_n \end{pmatrix}$$

where $x_i = \frac{i}{2\pi}\lambda_i$ and λ_i are eigenvalues of A, then

$$\det(I + \frac{i}{2\pi}A) = \prod_{i=1}^{n}(1+x_i) = 1 + \sum x_i + \sum_{i \neq j} x_i x_j + \cdots + \prod_{i=1}^{n} x_i$$]

Sometimes more convenient to define the **Chern characters**:

$$ch_k(E) = \frac{1}{k!}\,\mathrm{Tr}(\frac{i}{2\pi}F)^k \qquad k = 0, 1, 2, \cdots.$$

Also, the **Total Chern character** is defined by

$$\begin{aligned} ch(E) &= \sum_{k=0}^{\infty} ch_k(E) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathrm{Tr}(\frac{i}{2\pi}F)^k \\ &= \mathrm{Tr}(\,\sum_{k=0}^{\infty} \frac{(\frac{i}{2\pi}F)^k}{k!}\,) \\ &= \mathrm{Tr}(\,\exp(\frac{i}{2\pi}F)\,). \end{aligned}$$

Property 1. $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2) \qquad (\ in\ \ H^*(B, \mathbb{R})\ ).$

Proof: Let's pick connections D_i on E_i, $i = 1, 2$. Then we construct $D = D_1 \oplus D_2$. With respect to local frames, we get

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \& \quad F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}.$$

Therefore

$$\begin{aligned}
c(E) &= c(E_1 \oplus E_2)\\
&= \det\left(I + \frac{i}{2\pi}F\right)\\
&= \det\left(\begin{bmatrix} I_1 + \frac{i}{2\pi}F_1 & 0 \\ 0 & I_2 + \frac{i}{2\pi}F_2 \end{bmatrix}\right)\\
&= \det\left(I_1 + \frac{i}{2\pi}F_1\right) \cdot \det\left(I_2 + \frac{i}{2\pi}F_2\right)\\
&= c(E_1) \cdot c(E_2)
\end{aligned}$$

□

Corollary. *We can extract formulae for $c_j(E)$ from*

$$\sum_{j=0}^{n_1+n_2} c_j(E) = \left(\sum_{i=0}^{n_1} c_i(E_1)\right) \cdot \left(\sum_{k=0}^{n_2} c_k(E_2)\right).$$

For example,

$$\begin{aligned}
\deg 1: &\quad c_1(E) = c_1(E_2) + c_1(E_1)\\
\deg 2: &\quad c_2(E) = c_2(E_1) + c_1(E_1)c_1(E_2) + c_2(E_2)\\
&\vdots\\
\text{etc.}
\end{aligned}$$

Property 2. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$*, hence* $ch_k(E_1 \oplus E_2) = ch_k(E_1) + ch_k(E_2)$.

Proof: Use $D = D_1 + D_2$ and

$$\begin{aligned}
\mathrm{Tr}(\exp(\frac{i}{2\pi}F)) &= \mathrm{Tr}\left(\begin{bmatrix} \exp(\frac{i}{2\pi}F_1) & 0 \\ 0 & \exp(\frac{i}{2\pi}F_2) \end{bmatrix}\right)\\
&= \mathrm{Tr}(\exp(\frac{i}{2\pi}F_1)) + \mathrm{Tr}(\exp(\frac{i}{2\pi}F_2)).
\end{aligned}$$

□

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