THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 32

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Characteristic Classes for Real Bundles

Before we return to *Characteristic Classes* for *real* bundles, we go back to *complex* bundles: we had *Chern Classes* defined by the generating function

$$c(A) = \det(I + \frac{i}{2\pi}A) = \prod_{i=1}^{n}(1 + x_i) \quad (\text{where } x_i \text{ are the eigenvalues of } \frac{i}{2\pi}A) \ .$$

Chern Characters defined by the generating function

$$ch(A) = \mathrm{Tr}\left(\exp(\frac{i}{2\pi}A)\right) = \sum_{i=1}^{n} \exp(x_i) \ .$$

Another important characteristic class is the **Todd class**

$$Td(A) = \prod_{j=1}^{n} \frac{x_j}{1 - e^{-x_j}} \ .$$

Note.

- When applied to $A = F =$ curvature, series terminates at terms of deg $= \frac{\dim B}{2}$.
- For a complex bundle E, $Td(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)) + \cdots$.
- Appears in Riemann-Roch Theorem.

For a holomorphic bundle (complex bundle such that E is a complex manifold and π is holomorphic) over a complex manifold $E \xrightarrow{\pi} X$, we have an integer

$$\begin{aligned} \chi(E) &= \text{bundle version of } \textbf{Euler characteristic} \text{ of a manifold.} \\ &= \textbf{index} \text{ of a differential operator.} \end{aligned}$$

The Riemann-Roch Theorem relates this (global, topological) quantity $\chi(E)$ to characteristic classes:

$$\chi(E) = \int_X Td(T'X) \wedge ch(E) \ .$$

This is an example of an **Index Theorem** which relates the (integer valued) index of a differential operator $\Omega^0(E_1) \to \Omega^0(E_2)$ to $\int_B$(characteristic classes of TB and bundles).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

If X is a *Riemann surface* ($\dim_{\mathbb{C}} X = 1$), then $\text{Rank}(T'X) = 1$ and $Td(T'X) = 1 + \frac{1}{2}c_1(T'X)$. So

$$Td(T'X) \wedge ch(E) = (1 + \frac{1}{2}c_1(T'X)) \wedge (r + c_1(E)) \ .$$

Thus

$$\begin{aligned}\int_X Td(T'X) \wedge ch(E) &= \int_X (1 + \frac{1}{2}c_1(T'X)) \wedge (r + c_1(E)) \\ &= \int_X c_1(E) + \frac{r}{2}\int_X c_1(T'X) \ .\end{aligned}$$

Fact / Definition.

- $\int_X c_1(E) \equiv \deg(E)$.
- $\deg(T'X) = 2 - 2g$, where $g =$ genus of Riemann surface X.

So the Riemann-Roch Theorem:

$$\chi(E) = \deg(E) + \text{Rank}(E)(1 - g) \ .$$

Back to the Characteristic Classes for Real Bundles

Recall the First Approach from last time:

Given a real bundle $E_{\mathbb{R}}$ with fiber $\mathbb{R}^n$ and $\text{GL}(n, \mathbb{R})$, we can define a complex bundle $E_{\mathbb{C}} = E_{\mathbb{R}} \otimes \mathbb{C}$ with fiber $\mathbb{C}^n$ and $\text{GL}(n, \mathbb{C})(\hookleftarrow \text{GL}(n, \mathbb{R}))$.

Exercise 1. Show that $E_{\mathbb{C}} = E_{\mathbb{R}} \oplus E_{\mathbb{R}}$ as real bundle.

Hint: Look in detail at $\text{GL}(n, \mathbb{R}) \hookrightarrow \text{GL}(n, \mathbb{C}) \hookrightarrow \text{GL}(2n, \mathbb{R})$. □

Note. Given any complex bundle, we can look at underlying *real* bundle uning $\mathbb{C}^n \cong \mathbb{R}^{2n}$ to realize fibers as real and $\text{GL}(n, \mathbb{C}) \hookrightarrow \text{GL}(2n, \mathbb{R})$ to obtain real transition functions.

Therefore we can define

$$c_k(E_{\mathbb{R}}) \equiv c_k(E_{\mathbb{C}}) \ .$$

Note. For k odd, $c_k(E_{\mathbb{R}}) = 0$.

Proof: Given a complex bundle $E_{\mathbb{C}}$, we can define $\overline{E_{\mathbb{C}}}$. If $g_{\alpha\beta} \in \text{GL}(n, \mathbb{C})$ are transition functions for $E_{\mathbb{C}}$, then $\overline{g_{\alpha\beta}} \in \text{GL}(n, \mathbb{C})$ are transition functions for $\overline{E_{\mathbb{C}}}$. We now use the following key facts:

Key Facts.

(1) If $E_{\mathbb{C}} = E_{\mathbb{R}} \otimes \mathbb{C}$, then $\overline{E_{\mathbb{C}}} = E_{\mathbb{C}}$.
(2) $c_k(E_{\mathbb{C}}) = (-1)^k c_k(\overline{E_{\mathbb{C}}})$, for any complex bundle $E_{\mathbb{C}}$.

Hint: Given a connection D on $E_{\mathbb{C}}$, we can get a connection $\overline{D}$ on $E_{\mathbb{C}}$ such that $F_{\overline{D}} = \overline{F_D}$, then $i F_{\overline{D}} = -(\overline{i F_D})$. □

Thus $c_k(E_{\mathbb{C}}) = (-1)^k c_k(\overline{E_{\mathbb{C}}}) = (-1)^k c_k(E_{\mathbb{C}})$. □

So we can get $c_{2k}(E_{\mathbb{R}})$ in $H^{4k}(B, \mathbb{R})$.

Second Approach: Look at $\text{GL}(n, \mathbb{R})$ invariant polynomials on $\text{Mat}_n(\mathbb{R})$.

Easier: Look at $\text{O}(n)$-invariant polynomials on $\text{Lie}(\text{O}(n)) = \{ A | A^t + A = 0 \}$.

Why can we always do this: Fix a Riemann metric on $E_{\mathbb{R}}$ and use orthonormal frames and orthogonal connections, then the curvature F is $\text{Lie}(\text{O}(n))$-valued and the transition functions are $\text{O}(n)$-valued. Hence give any $\text{O}(n)$-invariant $P : \text{Lie}(\text{O}(n)) \to \mathbb{R}$, we can get $[P(\frac{1}{2\pi}F)] \in H^*(B, \mathbb{Z})$.

So what can we say about such P?

Fact. $A + A^t = 0$, A cannot be diagonalized:

Since

$$\begin{cases} A + A^t = 0 \\ A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \end{cases} \quad \Rightarrow \quad \lambda_i = 0 \; !$$

But A is conjugate (via $T \in \mathrm{O}(n)$) to

$$\underset{n = 2k}{\begin{bmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_k \\ & & & -\lambda_k & 0 \end{bmatrix}} \quad \text{or} \quad \underset{n = 2k+1}{\begin{bmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \lambda_k & \\ & & & -\lambda_k & 0 & \\ & & & & & 0 \end{bmatrix}} .$$

Thus we only need to consider P's on such A. Write

$$P \begin{bmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & & \ddots \end{bmatrix} = P(\lambda_1, \cdots, \lambda_n) \; .$$

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