THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 33

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Characteristic Classes for Real Bundles

We defined characteristic classes for a real bundle with a metric by using an $\mathrm{O}(n)$ invariant polynomial $P : \mathrm{Lie}(\mathrm{O}(n)) \to \mathbb{R}$ evaluated on $\frac{1}{2\pi}F$, where F is the curvature of any orthogonal connection with respect to local orthonormal frames.

We need to consider the possibilities for P. The key fact that we use is that if $A \in \mathrm{Lie}(\mathrm{O}(n))$, i.e, $A + A^t = 0$, then A is conjugate to a matrix of type I (if n is even)

$$\begin{pmatrix} A_1 & & \\ & \dots & \\ & & A_k \end{pmatrix}$$

or type II(if n is odd)

$$\begin{pmatrix} A_1 & & & \\ & \dots & & \\ & & A_k & \\ & & & 0 \end{pmatrix}$$

where A_i is the 2×2 block

$$A_i = \begin{pmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{pmatrix}$$

Specifically, there is a $T \in \mathrm{O}(n)$ such that $T^{-1}AT$ is shown as above. Since P is invariant, $P(A) = P(T^{-1}AT)$. Hence, we can express P as a function of the λ_i, i.e, $P(A) = P(\lambda_1, \dots, \lambda_n)$.

Claim.
(i) P is invariant under the action $\lambda_i \mapsto -\lambda_i$.
(ii) P is symmetric in $\lambda_1, \dots, \lambda_n$.

Corollary. *P is symmetric in $\{\lambda_i^2\}$.*

Proof: To switch λ_i with $-\lambda_i$, use the diagonal block matrix

$$T = \begin{pmatrix} I_2 & & & & \\ & \dots & & & \\ & & J & & \\ & & & \dots & \\ & & & & I_2 \end{pmatrix}$$

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Where J is the 2×2 block matrix, set in the (i,i) slot,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and I_2 is the 2×2 identity matrix.

To permute the $\{\lambda_1, \ldots, \lambda_n\}$, say $\lambda_3 \mapsto \lambda_i$, use a 2×2 block version of a permutation matrix,

$$T = \begin{pmatrix} & & I_2 & & \\ & I_2 & & & \\ I_2 & & & & \\ & & & I_2 & \\ & & & & \ldots \end{pmatrix} \quad \square$$

Corollary. *Any such P can be written as*

$$P(\lambda_1, \ldots, \lambda_k) = Q(\sigma_1(\lambda_1^2, \ldots, \lambda_k^2), \ldots, \sigma_{m/2}(\lambda_1^2, \ldots, \lambda_k^2))$$

where Q is a polynomial, m is the degree of P, and the σ_i are the elementary symmetric polynomials.

Definition 1. Let P_k be the polynomial $P_k(A) = \sigma_k(\lambda_1^2, \ldots, \lambda_k^2)$ where A is conjugate to one of the above matrices. We define the *k-th Pontrjagin class* of $E_{\mathbb{R}} \to B$ to be $[P_k(\frac{1}{2\pi}F] \in H^{4k}(B;\mathbb{R})$ where F is the curvature of any connection.

Exercise 1. Check that the same proof as before (i.e, as used for the Chern classes of a complex bundle) shows that $[P_k(\frac{1}{2\pi}F]$ defines a characteristic class.

Exercise 2. Prove that if $A + A^t = 0$, then we have that $\det(I + \frac{1}{2\pi}A) = \sum P_k(\frac{1}{2\pi}F)$.

The Relationship Between $P_k(E_{\mathbb{R}})$ and $C_k(E\mathbb{C})$

Proposition. $P_k(E_{\mathbb{R}}) = (-1)^k C_{2k}(E_{\mathbb{C}})$.

Proof: The complexification of $E_{\mathbb{R}}$ is defined, as before, to be $E_{\mathbb{C}} = E_{\mathbb{R}} \otimes \mathbb{C}$. Fix a metric on $E_{\mathbb{R}}$. Say $\{e_i^\alpha\}$ are orthonormal frames for $E_{\mathbb{R}}$ with $\mathrm{O}(n)$-valued transition functions $g_{\alpha\beta}$. Under the inclusion $\mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{C})$, we have that $\mathrm{O}(n) \to \mathrm{U}(n)$. So, we can think of $\{e_i^\alpha\}$ as unitary frame for $E_{\mathbb{C}}$.

Suppose that $D = d + A$ is the local description of an orthogonal connection on $E_{\mathbb{R}}$ (so that A is $\mathrm{Lie}(\mathrm{O}(n))$-valued). After we include $\mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{C})$, A can be thought of as skew hermitian, e.g, $\mathrm{Lie}(\mathrm{U}(n))$-valued. Denote this by $A_{\mathbb{C}}$. So, $d + A_{\mathbb{C}}$ is now the local expression for a unitary connection on $E_{\mathbb{C}}$.

Let $F_{\mathbb{R}}$ be the curvature of A. Once again, $F_{\mathbb{R}}$ is $\mathrm{Lie}(\mathrm{O}(n))$-valued. Denote the curvature of $A_{\mathbb{C}}$ by $F_{\mathbb{C}}$. Again, $F_{\mathbb{C}}$ is $\mathrm{Lie}(\mathrm{U}(n))$-valued. Over $\mathrm{O}(n)$, $F_{\mathbb{R}}$ is conjugate to a block matrix of either type I or type II. Over $\mathrm{U}(n)$, $F_{\mathbb{C}}$ can be diagonalized with complex Eigenvalues μ_i. Now,

$$P_k(E_{\mathbb{R}}) = \sigma_k(\frac{\lambda_1^2}{2\pi}, \ldots, \frac{\lambda_m^2}{2\pi})$$

and

$$C_k(\frac{i}{2\pi}\mu_1, \ldots, \frac{i}{2\pi}\mu_m)$$

We can actually do better for $F_{\mathbb{C}}$. Observe that $F_{\mathbb{C}}$ is conjugate to a matrix of the form

$$\begin{pmatrix} i\lambda_1 & & & & \\ & -i\lambda_1 & & & \\ & & \cdots & & \\ & & & i\lambda_m & \\ & & & & -i\lambda_m \end{pmatrix}$$

[To see this, in the rank 2 case, we can simply have that

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

is U(2) conjugate to

$$\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix}$$

via the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}]$$

Thus,

$$\begin{aligned} C_{2k}(E_{\mathbb{C}}) &= \sigma_{2k}(\frac{i}{2\pi}(i\lambda_1), \frac{i}{2\pi}(i\lambda_1), \ldots, \frac{i}{2\pi}(i\lambda_m), \frac{i}{2\pi}(-i\lambda_m)) \\ &= \sigma_{2k}(-\frac{1}{2\pi}\lambda_1, \frac{1}{2\pi}\lambda_1, \ldots, -\frac{1}{2\pi}\lambda_m, \frac{1}{2\pi}\lambda_m) \end{aligned}$$

The proof reduces to showing the following lemma from the theory of symmetric polynomials(which is left as an exercise):

$$(-1)^k\sigma_k(x_1^2, \ldots, x_m^2) = \sigma_{2k}(x_1, -x_1, \ldots, x_m, -x_m)$$

□

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