THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 34

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For a real bundle we defined the *k-th Pontrjagin class*, $P_k(E_{\mathbb{R}}) \in H^{4k}(B;\mathbb{R})$. We saw that $P_k(E_{\mathbb{R}}) = (-1)^k C_{2k}(E_{\mathbb{C}})$ where $E_{\mathbb{C}}$ was the complexification of $E_{\mathbb{R}}$, $E_{\mathbb{C}} = E_{\mathbb{R}} \otimes \mathbb{C}$. If we start with a complex bundle, E_c, then there is a natural underlying real bundle, E_r, obtained by forgetting the holomorphic structure. Specifically, we identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and we embed $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$. The inclusion for the case where $n=1$ is

$$(z) \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

where $z = x+iy$. The higher dimensional case is similar. If

$$E_c = (\coprod U_\alpha \times \mathbb{C}^n)/g_{\alpha\beta},$$

then

$$E_r = (\coprod U_\alpha \times \mathbb{R}^{2n})/g^r_{\alpha\beta}$$

where the $g^r_{\alpha\beta}$ are the real analogues of $g_{\alpha\beta}$ obtained from the inclusion $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$. As for any real bundle, we can define the Pontrjagin classes for E_r.

Question: How are $P_k(E_r)$ and $C_k(E_c)$ related?

Observe that E_c is not $E_r \otimes \mathbb{C}$, for E_c has complex rank n while $E_r \otimes \mathbb{C}$ has complex rank $2n$.

Exercise 1. Show that $E_r \otimes \mathbb{C} \cong E_c \oplus \bar{E}_c$ as complex bundles.

We thus get $P_k(E_r) = (-1)^k c_{2k}(E_r \otimes \mathbb{C}) = (-1)^k c_{2k}(E_c \oplus \bar{E}_c)$. It follows that

$$\sum P_k(E_r)(-1)^k = \sum c_{2k}(E_c \oplus \bar{E}_c)$$

Exercise 2. Show that $c_{2k+1}(E \oplus \bar{E}) = 0$. (*Hint:* $c_l(\bar{E}) = (-1)^l c_l(E)$)

Using this, we can write

$$\sum P_k(E_r)(-1)^k = \sum c_k(E_c \oplus \bar{E}_c) = c(E_c \oplus \bar{E}_c) = c(E_c)c(\bar{E}_c)$$

This answers our question and shows how the $c_k(E_c)$ determine the $P_k(E_r)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Note. The inclusion $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ has its image in the component of $\mathrm{GL}(2n,\mathbb{R})$ consisting of invertible matrices with positive determinant, $\mathrm{GL}^+(2n,\mathbb{R})$. Hence, the image of $\mathrm{U}(n)$ lies in $\mathrm{SO}(2n)$.

Example 1. $\mathrm{U}(1)$ embeds in $\mathrm{SO}(n)$ by

$$e^{i\theta} \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

So, if

$$E_c = (\coprod U_\alpha \times \mathbb{C}^n)/g_{\alpha\beta}$$

and

$$E_r = (\coprod U_\alpha \times \mathbb{R}^{2n})/g^r_{\alpha\beta}$$

then $\det(g^r_{\alpha\beta}) > 0$. Similarly, if we put a metric on E_c, then $g_{\alpha\beta}$ is $\mathrm{U}(n)$-valued and $g^r_{\alpha\beta}$ is $\mathrm{SO}(n)$-valued.

Definition 1. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible with positive determinant, call T an *orientation preserving transformation*. Equivantly, if $\{e_i\}$ is a frame in $\mathbb{R}^n$, then $\{e_i\}$ and $\{Te_i\}$ have the same orientation. If $\det(g_{\alpha\beta} > 0$, then we can consistently assign an orientation to the fibers of E by using the trivializations of the bundle, e.g, if $\psi_b : E_b \to \mathbb{R}^n$ is the isomorphism with $b \in U_\alpha$, then call $\{\psi_\alpha^{-1} e_i\}$ the *positively oriented frame* for E_b, where $\{e_i\}$ is a standard frame for $\mathbb{R}^n$.

Definition 2. Call a bundle E *orientable* if this is possible, i.e, if we can find transition functions $\{g_{\alpha\beta}\}$ with $\det(g_{\alpha\beta}) > 0$.

This is equivalent to the claim that we can pick frames where the transition functions are $\mathrm{SO}(n)$-valued.

Corollary. *E_r is always an orientable bundle.*

Orientable Bundles

Suppose that $E \to B$ is an orientable bundle; so, the transition functions can be chosen to be $\mathrm{SO}(n)$-valued. In this case, when we check the invariance of a polynomial, $P : \mathrm{Lie}(\mathrm{O}(n)) \to \mathbb{R}$, we need only check the invariance under conjugation by $T \in \mathrm{SO}(n)$.

Note. Not all bundles, E, are orientable. How to determine whether a given bundle is orientable is an interesting question, to which we shall return in the coming lectures.

Question: Are there any P which are $\mathrm{SO}(n)$ invariant but not $\mathrm{O}(n)$ invariant? If yes, then these define new characteristic classes for orientable bundles which nonorientable bundles will not have.

Answer: As we shall see, when the rank of the bundle is odd, the answer is no. When the rank is even, there is a new class, called the *Euler class*, $e(E)$.

We saw in the previous lecture that A is $\mathrm{O}(n)$ equivalent (under conjugation) to a matrix of one of two types and that $P(A) = P(\lambda_1, \ldots, \lambda_n)$ where P is an $\mathrm{O}(n)$ invariant polynomial. Furthermore, we saw that this was symmetric in the $\{\lambda_i^2\}$.

Case n=2

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

via

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

But, $\det(T) = -1$ and so $T \in \mathrm{O}(n)$, $T \notin \mathrm{SO}(n)$.

Case n=3

$$\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

via

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

If we choose -1, then $T \in \mathrm{SO}(n)$.

In general, if n is odd, then $\mathrm{SO}(n)$ invariance implies the symmetry under $\{\lambda_i^2\}$, which implies $\mathrm{O}(n)$ invariance. If n is even, this won't happen.

Claim. *If P is $\mathrm{SO}(n)$ invariant, then we can write P as $P = P_0 + P_1$ where*
(1) P_0 is $\mathrm{O}(n)$ invariant.
(2) $P_1(gAg^{-1}) = (det(g))P_1(A)$ for all $g \in \mathrm{O}(n)$.

Proof: Pick $g_0 \in \mathrm{O}(n) - \mathrm{SO}(n)$. Write

$$P(A) = \frac{P(A) + P(g_0 A g_0^{-1})}{2} + \frac{P(A) - P(g_0 A g_0^{-1})}{2}$$

Then

$$P_0(A) = \frac{1}{2}(P(A) + P(g_0 A g_0^{-1}))$$

and

$$P_1(A) = \frac{1}{2}(P(A) - P(g_0 A g_0^{-1}))$$

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