THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 35

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Suppose $E \to B$ is a rank $n = 2m$ bundle and that P is a $\mathrm{SO}(n)$ invariant polynomial on $\{A + A^t = 0\} = \mathrm{Lie}(\mathrm{O}(n))$. Then $P(A) = P_0(A) + P_1(A)$ where P_0 is a fully $\mathrm{O}(n)$ invariant polynomial and P_1 is a $\mathrm{SO}(n)$ invariant polynomial. We saw that $\det(g)P_1(A) = P_1(gAg^{-1})$ for $g \in \mathrm{O}(n)$.

Example 1. Define $e(A)$ as follows. Fix an oriented, orthonormal basis for $\mathbb{R}^{2n}$, say $\{e_i\}$. Let $Ae_i = A_{ji}e_j$. Define

$$\alpha(A) = \sum_{i<j} A_{ij} e_i \wedge e_j \in \bigwedge^2(\mathbb{R}^{2m})$$

[If A is a matrix of standard type I, then a direct calculation shows that

$$\alpha(A) = \sum_{i=1}^{m} x_i e_{2i-1} \wedge e_{2i}]$$

Set

$$e(A) = \frac{1}{m!}(\alpha(A)^m, e_1 \wedge \cdots \wedge e_{2m})$$

where $(\quad,\quad)$ is the inner product on $\bigwedge^{2m}(\mathbb{R}^{2m})$. [For A as above we get that

$$\alpha(A)^m = m!(x_1 x_2 \dots x_m e_1 \wedge \cdots \wedge e_{2m})$$

So, $e(A) = x_1 \dots x_m$.]

Exercise 1. Show that

(1) $e(gAg^{-1}) = e(A)\det(g)$ for $g \in \mathrm{O}(n)$.
(2) $e^2(A) = \det(A)$.

Fact: Any $\mathrm{SO}(n)$ invariant polynomial can be written as $P(A) = e(A)\hat{P}(A)$ where $\hat{P}$ is a $\mathrm{O}(n)$ invariant polynomial.

Definition 1. Given an oriented real vector bundle $E \to B$ of rank $n = 2m$, we define the *Euler class* by $e(\frac{1}{2\pi}F)$ where F is the curvature of any orthogonal connection with respect to an oriented, orthonormal frame.

Note.

(1) If the rank of E is $n = 2m$, then $e(E) \in H^{2m}(B;\mathbb{R})$. In fact, $e(E) \in H^{2m}(B;\mathbb{Z})$.
(2) The Euler class distributes nicely across sums. That is, $e(E_1 \oplus E_2) = e(E_1)e(E_2)$.

Special Case: Suppose that $E = E_r$ is the underlying real bundle of a complex bundle E_c. From previous lectures, E_r is orientable and if E_c has complex dimension m, then E_r has real dimension $2m$.

Question: How is $e(E_r)$ related related to the Chern classes of E_c?

Clue: $e^2(A) = \det(A)$. Also, $\det(I + A) = 1 + \cdots + \det(A)$. So, $e^2(E_r) = P_m(E_r)$. Also,

$$\sum_{k=0}^{m} P_k(E_r)(-1)^k = (1 + c_1(E_c) + \cdots + c_m(E_c))(1 - c_1(E_c) + \cdots + (-1)^m c_m(E_c))$$

This implies that $P_m(E_r)(-1)^m = (-1)^m c_m^2(E_c)$. So, $e^2(E_r) = c_m^2(E_c)$. In fact, we shall see that $e(E_r) = c_m(E_c)$, where E_r has a standard orientation induced by E_c.

The orientation on $\mathbb{R}^{2m}$ induced by an orientation from $\mathbb{C}^m$ is obtained as follows. Choose a basis $\{e_a\}_{a=1}^m$ for $\mathbb{C}^m$ over $\mathbb{C}$. Then $\{e_a, ie_a\}_{a=1}^m$ is a basis for $\mathbb{R}^{2m}$. The orientation on $\mathbb{R}^{2m}$ is obtained by declaring that this basis is positively oriented.

Under this choice of frames, the inclusion $\mathrm{GL}(m,\mathbb{C}) \to \mathrm{GL}(2m,\mathbb{R})$ yields

$$\begin{pmatrix} \frac{i\lambda_1}{2\pi} & & \\ & \cdots & \\ & & \frac{i\lambda_m}{2\pi} \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\frac{\lambda_1}{2\pi} & & & \\ \frac{\lambda_1}{2\pi} & 0 & & & \\ & & \cdots & & \\ & & & 0 & -\frac{\lambda_m}{2\pi} \\ & & & \frac{\lambda_m}{2\pi} & 0 \end{pmatrix}$$

It follows from this that

$$c_m(E_c) = \prod_{i=1}^{m} -\frac{\lambda_i}{2\pi} = e(E_r)$$

Note. The signs have been chosen so that this holds. If we change the orientation of E, then $e(E)$ will change signs.

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