THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 37

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Class Notes From Math 433

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April 29, 1998

Čech Cohomology and Orientability

Question: Given a vector bundle $E = \coprod U_\alpha \times \mathbb{R}^n / g_{\alpha\beta}$, $g_{\alpha\beta} \in \mathrm{O}(n)$. When can local frames be chosen such that $g_{\alpha\beta} \in \mathrm{SO}(n)$?

Last time: Say we **can** find local **gauge** transformations $\lambda_\alpha : U_\alpha \to \mathrm{O}(n)$ such that under

$$E|_{U_\alpha} \overset{\psi_\alpha}{\underset{\simeq}{\longrightarrow}} U_\alpha \times \mathbb{R}^n \overset{1\times\lambda_\alpha}{\longrightarrow} U_\alpha \times \mathbb{R}^n \,,$$

new transition functions $\widetilde{g}_{\alpha\beta} = \lambda_\alpha \circ g_{\alpha\beta} \circ \lambda_\beta^{-1}$ satisfy

$$\det \widetilde{g}_{\alpha\beta} = 1 \ .$$

We can reformulate this using

$$\begin{aligned} \det(g_{\alpha\beta}) &= e^{i\pi u_{\alpha\beta}} \\ \det(\lambda_\alpha) &= e^{i\pi v_\alpha} \end{aligned} \qquad u_{\alpha\beta}\ ,\ v_\alpha \ \in \{0,1\} = \mathbb{Z}_2 \,.$$

Then

$$\begin{aligned} \text{the cocycle condition} &\Longrightarrow u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = 0 \qquad \text{(closed)}\ , \\ \text{and the condition} \quad \det \widetilde{g}_{\alpha\beta} = 1 &\Longrightarrow u_{\alpha\beta} = v_\beta - v_\alpha \qquad \text{(exact)}\ . \end{aligned}$$

Cohomology Intepretation (using Čech cohomology with $\mathbb{Z}_2$ coefficients)

- Fix "suitable" cover $\{U_\alpha\}_{\alpha \in I}$ for B, where "suitable" means all intersections are contractible and connected.
- Define a 0-**cochain** to be

$$g^{(0)} = \{\, g_\alpha \in \mathbb{Z}_2 \mid \alpha \in I \,\} \quad (\text{so } g^{(0)} \text{ defines a map } g^{(0)} : I \to \mathbb{Z}_2)\ .$$

- Define a 1-**cochain** to be

$$g^{(1)} = \{\, g_{\alpha\beta} \in \mathbb{Z}_2 \mid \alpha \neq \beta,\ U_\alpha \cap U_\beta \neq \emptyset,\ g_{\alpha\beta} = -g_{\beta\alpha} (= g_{\beta\alpha} \text{ in } \mathbb{Z}_2!) \,\}.$$

- Define a **j-cochain** to be

$$g^{(j)} = \{\, g_{\alpha_0\cdots\alpha_j} \in \mathbb{Z}_2 \mid \forall i \neq j,\ \alpha_i \neq \alpha_j,\ \bigcap_{i=0}^{j} U_{\alpha_i} \neq \emptyset,\ g_{\alpha_0\cdots\alpha_j} \text{ antisymmetric w.r.t. indices} \,\}\ .$$

- Define

$\check{C}^{(j)}$ = Group of all j-cochains with group operation $g^{(j)}+f^{(j)} = \{\, (g+f)_{\alpha_0\cdots\alpha_j} = g_{\alpha_0\cdots\alpha_j} + f_{\alpha_0\cdots\alpha_j} \,\}$.

- Define $\delta : \check{C}^{(j)} \to \check{C}^{(j+1)}$ by

$$(\delta g^{(j)})_{\alpha_0\cdots\alpha_{j+1}} := \sum_{i=0}^{j+1} (-1)^i g^{(j)}_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{j+1}}\ ,$$

where the "$\hat{}$" on $\hat{\alpha}_i$ means "delete from the list of indices". For example,

$$\begin{array}{ll} \delta : C^{(0)} \to C^{(1)} & (\delta g^{(0)})_{\alpha\beta} \ = g^{(0)}_{\beta} - g^{(0)}_{\alpha} \\ \delta : C^{(1)} \to C^{(2)} & (\delta g^{(1)})_{\alpha\beta\gamma} = g^{(1)}_{\beta\alpha} - g^{(1)}_{\alpha\gamma} + g^{(1)}_{\alpha\beta}\ . \end{array}$$

Key Facts: In

$$\check{C}^{(0)} \xrightarrow{\delta^{(0)}} \check{C}^{(1)} \xrightarrow{\delta^{(1)}} \check{C}^{(2)} \xrightarrow{\delta^{(2)}} \cdots\ .$$

(a) δ is a group homomorphism.
(b) $\delta^2 = 0$.

Exercise 1. Prove the above key facts.

Thus $\operatorname{Im}\delta^{(p)} \subseteq \operatorname{Ker}\delta^{(p+1)}$ and we can form

$$\check{H}^{(p)}\left(\{U_\alpha\}_{\alpha\in I}, \mathbb{Z}_2\right) = \frac{\operatorname{Ker}\delta^{(p)}}{\operatorname{Im}\delta^{(p-1)}}\ .$$

Facts:
(1) Take $\check{H}^{(0)} = \operatorname{Ker}\delta^{(0)}$.
(2) For suitable covers, this result is independent of $\{U_\alpha\}_{\alpha\in I}$.
(3) $\check{H}^{(p)}(B,\mathbb{Z}_2) \cong H^p(B,\mathbb{Z}_2)$.

Back to $\{u_{\alpha\beta}\}$ and $\{v_\alpha\}_{\alpha\in I}$

- Clearly $\{u_{\alpha\beta}\}$ defines a Čech 1-cochain.
- $u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = 0$ says $\delta^{(1)}(\{u_{\alpha\beta}\}) = 0$. (since $g_{\alpha\beta} = g^{-1}_{\beta\alpha}$, we get $u_{\gamma\alpha} = -u_{\alpha\gamma}$ etc.)
- Hence $\{u_{\alpha\beta}\}$ defines elements in $H^{(1)}(B,\mathbb{Z}_2) = H^1(B,\mathbb{Z})$. [This is called $w_1(E)$, the **first Stiefel-Whitney class**.]
- $\{v_\alpha\}_{\alpha\in I}$ defines $v^{(0)} \in \check{C}^{(0)}$ and $u_{\alpha\beta} = v_\beta - v_\alpha$ says $\{u_{\alpha\beta}\} = \delta v^{(0)}$, i.e., $w_1(E) = 0$.

Conclusion. *E is orientable $\Longrightarrow w_1(E)$.*

Remark. The converse holds too:

Consider a vector bundle $E = \coprod U_\alpha \times \mathbb{R}^n / g_{\alpha\beta}$. Define the Čech cochain $u^{(1)} = \{u_{\alpha\beta}\}$ as above, i.e., $\det g_{\alpha\beta} = e^{i\pi u_{\alpha\beta}}$. As above, this defines $w_1(E) = [u^{(1)}] \in H^1(B, \mathbb{Z}_2)$.

Say $w_1(E) = 0$, i.e., $[\{u_{\alpha\beta}\}] = 0 \in \check{H}^{(1)}(B, \mathbb{Z}_2)$. It follows that we can find $\{v_\alpha\}$ such that $u_{\alpha\beta} = v_\beta - v_\alpha$.

Claim. *Given $\{v_\alpha\}$ with $v_\alpha : U_\alpha \to \mathbb{Z}_2$, we can find $\lambda_\alpha : U_\alpha \to \mathrm{O}(n)$ such that $\det \lambda_\alpha = e^{i\pi v_\alpha}$.*

Exercise 2. Prove the above claim.

It follows from this, using λ_α to adjust local trivialization, that we get $\widetilde{g}_{\alpha\beta} \in \mathrm{SO}(n)$. That is, E is orientable.

We have thus proved:

Proposition. *E is orientable $\Longleftrightarrow w_1(E) = 0$.*

Higher Stiefel Whitney Classes and Obstructions

- Will see that $w_1(E)$ is not the only $\mathbb{Z}_2$-characteristic class. Hence $w_2(E) \in H^2(B, \mathbb{Z}_2)$ etc.
- $w_2(E)$ is also an obstruction — to the existence of a **Spin structure**.

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