THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 39

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Class Notes From Math 433

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May 4, 1998

Spin Structures

We had a double cover, $p : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. If

$$P_{\mathrm{SO}(n)} = \coprod U_\alpha \times \mathrm{SO}(n)/g_{\alpha\beta}$$

is a principal $\mathrm{SO}(n)$-bundle, then a spin structure for the bundle is

$$P_{\mathrm{Spin}(n)} = \coprod U_\alpha \times \mathrm{Spin}(n)/\tilde{g}_{\alpha\beta}$$

where $p\tilde{g}_{\alpha\beta} = g_{\alpha\beta}$. This yields a bundle map, $\rho : P_{\mathrm{Spin}(n)} \to P_{\mathrm{SO}(n)}$.

If $P_{\mathrm{SO}(n)}$ is the bundle of oriented frames for TM, then $P_{\mathrm{Spin}(n)}$ defines a spin structure on M. $TM = P_{\mathrm{SO}(n)} \times_\sigma \mathbb{R}^n$, where $\sigma : \mathrm{SO}(n) \to \mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$. Sections of TM are vector fields on M. Given $P_{\mathrm{Spin}(n)}$, denote the induced representation be $\tilde{\sigma} : \mathrm{Spin}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ and get $\mathbb{V} = P_{\mathrm{Spin}(n)} \times_{\tilde{\sigma}} \mathbb{R}^n$.

What is the relation between $\mathrm{SO}(n)$-representations and $\mathrm{Spin}(n)$-representations? Since by definition $\tilde{\sigma} = \sigma \circ p : \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{Aut}(\mathbb{R}^n)$, we have that every representation of $\mathrm{SO}(n)$ gives a representation of $\mathrm{Spin}(n)$.

Observe that $\tilde{\sigma}(-I) = \tilde{\sigma}(I) = I$. So, if a representation, $\tilde{\sigma}$ of $\mathrm{Spin}(n)$ is such that $\tilde{\sigma}(-I) = \tilde{\sigma}(I)$, then we can define a representation of $\mathrm{SO}(n)$ in the natural way. However, there are representations of $\mathrm{Spin}(n)$ which do not have this property. Infact, we can identify special irreducible representations. This leads to Spin bundles on M. Sections of these bundles are called *spinors*.

$\mathrm{Spin}_{\mathbb{C}}$ Structures

$\mathrm{Spin}(n)$ was defined by the exact sequence,

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

We now look to double cover $\mathrm{SO}(n) \times S^1 = \mathrm{SO}(n) \times \mathrm{U}(1)$. S^1 is naturally a double cover of S^1 under the map $Z \mapsto z^2$. We could thus look at the map, $\mathrm{Spin}(n) \times S^1 \to \mathrm{SO}(n) \times S^1$ given by the product of the two covering maps. However, this is a four-fold, rather than two-fold, cover of $\mathrm{SO}(n) \times S^1$. Observe that $\mathbb{Z}_2$ acts on both $\mathrm{Spin}(n)$ and S^1 (antipodal action). So we can examine the orbits of the action on $\mathrm{Spin}(n) \times S^1$,

$$\mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1 = \{[\tilde{g}, \lambda] : [\tilde{g}, \lambda] = [-\tilde{g}, -\lambda]\}$$

We thus make the following defintion,

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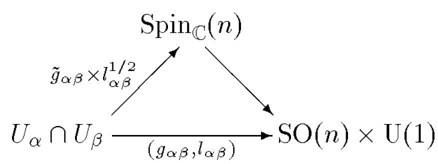
Definition 1. $\mathrm{Spin}_{\mathbb{C}}(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1$.

If we define $\mathrm{Spin}_{\mathbb{C}}(n) \to \mathrm{SO}(n) \times S^1$ by $[\tilde{g}, \lambda] \mapsto (g, \lambda)$, then $\mathrm{Spin}_{\mathbb{C}}$ is a double cover of $\mathrm{SO}(n) \times S^1$. We now want an analogue of the bundle, $P_{\mathrm{Spin}(n)} \to P_{\mathrm{SO}(n)}$. We have that $S^1 = \mathrm{U}(1)$ is the structure group of $P_{\mathrm{U}(1)}$ - a principal $\mathrm{U}(1)-$bundle (equivalently a complex line bundle). Starting with

$$P_{\mathrm{SO}(n)} \times L = \coprod U_\alpha \times (\mathrm{SO}(n) \times \mathrm{U}(1))/\{g_{\alpha\beta}, l_{\alpha\beta}\}$$

when can we lift to a $\mathrm{Spin}_{\mathbb{C}}$ bundle? Locally there is no problem to doing this,



just from basic covering space theory. However, can we pick the lifts $\tilde{g}_{\alpha beta}$ and $l_{\alpha\beta}^{1/2}$ in a way so that the cocyle condition is satisfied? That is, can we pick the lifts so that for all α, β, γ with $U_\alpha \cap U_\beta \cap U_g amma \neq \emptyset$,

$$[\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}, l_{\alpha\beta}^{1/2} l_{\beta\gamma}^{1/2} l_{\gamma\alpha}^{1/2}] = 1 \in \mathrm{Spin}_{\mathbb{C}}(n)$$

i.e, we must get $[1, 1]$ or $[-1, -1]$.

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