THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 4

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Recall from last time that given a vector bundle $\pi : E \to B$, say with fiber $\mathbb{R}^n$, we cover B with trivializing neighborhoods, U_α, with trivializations $\psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^n$, and obtain transition functions, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$ by $g_{\alpha\beta}(x) = \psi_\alpha(\psi_\beta^{-1}(x))$. These satisfy three properties:

(1) $g_{\alpha\alpha} = \mathrm{Id} \in \mathrm{GL}(n, \mathbb{R})$
(2) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
(3) (cocycle condition) $g_{\alpha\gamma} g_{\gamma\beta} g_{\beta\alpha} = \mathrm{Id}$

Claim. *Given a cover, $\{U_\alpha\}$ of B and $\{g_{\alpha\beta}\} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$, one for every nonemtpy intersection, such that the conditions 1,2,3 listed above hold, then there exists a vector bundle (unique up to bundle isomorphism) for which $\{g_{\alpha\beta}\}$ are the transition functions.*

Proof: The proof is a constructive one.

Step 1: For each U_α, define

$$Z = \coprod_{\alpha \in A} U_\alpha \times \mathbb{R}^n$$

Put the (product?) topology on Z.

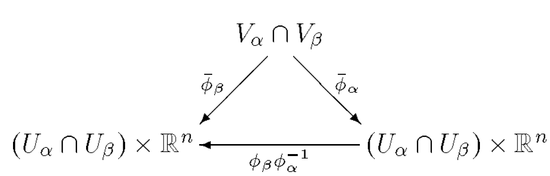
Step 2: Define an equivalence relation on Z as follows:

$$(x, v)_\alpha \sim (x', v')_\beta \text{ if and only if } x = x' \text{ and } v' = g_{\beta\alpha}(x)(v)$$

Exercise 1. Using conditions 1,2, and 3 above, show that this defines an equivalence relation.

Step 3: Define $E = Z/$. Put the quotient topology on E. We need to establish several properties of E. First, we need to show that since the $g_{\beta\alpha}$ are smooth maps, E is a smooth manifold. Second, the map $[(x, v)_\alpha] \mapsto x$ is a well defined map from E to B which makes this a rank n bundle. Third, we need to show that the set of transition functions for E is precisely $\{g_{\alpha\beta}\}$.

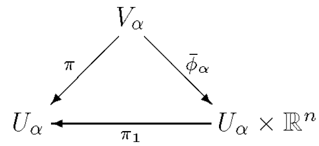
First, X is a smooth manifold if we can cover X with coordinate charts which are smoothly related. That is, whose transition functions are smooth. Assume that the cover $\{U_\alpha\}$ of B actually consists of coordinate patches with trivializations $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. Otherwise, we simply refine our cover so that it has this property. Assume that B has rank m. We need to define a collection, $\{V_\alpha\}$ of coordinate patches on E and maps, $\bar{\phi}_\alpha : V_\alpha \to \mathbb{R}^m \times \mathbb{R}^n$. Set $V_\alpha = [U_\alpha \times \mathbb{R}^n]$. Each V_α is open by the defintion of the quotient topology. Define $\bar{\phi}_\alpha$ by $[(x, v)_\alpha] \mapsto (\phi_\alpha(x), v)$. We have the following commutative diagram:



On the overlap, $\bar{\phi}_\beta \bar{\phi}_\alpha^{-1}$, has the following result,

$$(x, v) \mapsto \bar{\phi}_\beta([x, v]_\alpha) = \bar{\phi}_\beta([x, g_{\beta\alpha}(x)(v)]_\beta) = (x, g_{\beta\alpha}(x)(v))$$

Thus, on the overlap, the maps are smooth since the $g_{\beta\alpha}$ are. Since we now actually have a smooth manifold, we continue. We now need to examine the projection map, $\pi : E \to B$, defined above by $[(x,v)_\alpha] \mapsto x$. Locally, we have



Where π_1 is the natural projection. We have that $\pi = \pi_1 \circ \bar{\phi}_\alpha$ and so π is smooth (well defined, continuous, etc). By definition,

$$\bar{\psi}_\alpha^{-1}(\pi_1^{-1}(x)) = \{[(x,v)_\alpha] : v \in \mathbb{R}^n\} \subseteq \pi^{-1}(x)$$

Do we get anything else? Suppose that $[(x,v')_\beta]$ is in the fiber over x. Then $[(x,v')_\beta] = [(x, g_{\alpha\beta}(x)(v'))_\alpha]$. So, the full fiber over x is $\mathbb{R}^n$. The remainder of the proof is left as the exercise below. $\square$

Exercise 2. Show that, in fact, the above statements also yield the locally trivial condition and the transition properties

Recall that a bundle, $\pi : P \to B$ is a principal bundle $G-$bundle if $\pi^{-1}(x) \cong G$, where G is a Lie group, π is locally trivial, and there is a free right $G-$action on P which preserves the fibers. A map of two $G-$bundles, P and P' is a bundle map, $h : P \to P'$ which is $G-$equivariant; that is, $h(p \cdot g) = h(p) \cdot g$. The locally trivial condition says that the trivialization, $\psi_\alpha : P|U_\alpha \to U_\alpha \times G$ must be fiber preserving and $G-$equivariant; where the $G-$action on $U_\alpha \times G$ is trivial on U_α and group multiplication on the G factor. So, on $U_\alpha \cap U_\beta$, $\psi_\beta \psi_\alpha^{-1}$ has the form,

$$(x,g) = (x,1g) \mapsto (\psi_\beta \psi_\alpha^{-1}(x,1))g$$

Define $g_{\beta\alpha}(x) = (\psi_\beta \psi_\alpha^{-1}(x,1))$. So, we must have that $(x,g) \mapsto (x, g_{\beta\alpha}(x) \cdot g)$. Exactly as before, we define

$$P = (\coprod U_\alpha \times G)/\sim$$

where $\sim$ is defined, as before, by the $\{g_{\beta\alpha}\}$.

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