

**THE GEOMETRY OF VECTOR BUNDLES AND
AN INTRODUCTION TO GAUGE THEORY
LECTURE 4**

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Recall from last time that given a vector bundle $\pi : E \rightarrow B$, say with fiber \mathbb{R}^n , we cover B with trivializing neighborhoods, U_α , with trivialisations $\psi_\alpha : E|U_\alpha \rightarrow U_\alpha \times \mathbb{R}^n$, and obtain transition functions, $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{R})$ by $g_{\alpha\beta}(x) = \psi_\alpha(\psi_\beta^{-1}(x))$. These satisfy three properties:

- (1) $g_{\alpha\alpha} = \text{Id} \in GL(n, \mathbb{R})$
- (2) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
- (3) (cocycle condition) $g_{\alpha\gamma}g_{\gamma\beta}g_{\beta\alpha} = \text{Id}$

Claim. *Given a cover, $\{U_\alpha\}$ of B and $\{g_{\alpha\beta}\} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{R})$, one for every nonempty intersection, such that the conditions 1,2,3 listed above hold, then there exists a vector bundle (unique up to bundle isomorphism) for which $\{g_{\alpha\beta}\}$ are the transition functions.*

Proof: The proof is a constructive one.

Step 1: For each U_α , define

$$Z = \coprod_{\alpha \in A} U_\alpha \times \mathbb{R}^n$$

Put the (product?) topology on Z .

Step 2: Define an equivalence relation on Z as follows:

$$(x, v)_\alpha \sim (x', v')_\beta \text{ if and only if } x = x' \text{ and } v' = g_{\beta\alpha}(x)(v)$$

Exercise 1. Using conditions 1,2, and 3 above, show that this defines an equivalence relation.

Step 3: Define $E = Z / \sim$. Put the quotient topology on E . We need to establish several properties of E .

First, we need to show that since the $g_{\beta\alpha}$ are smooth maps, E is a smooth manifold. Second, the map $[(x, v)_\alpha] \mapsto x$ is a well defined map from E to B which makes this a rank n bundle. Third, we need to show that the set of transition functions for E is precisely $\{g_{\alpha\beta}\}$.

First, X is a smooth manifold if we can cover X with coordinate charts which are smoothly related. That is, whose transition functions are smooth. Assume that the cover $\{U_\alpha\}$ of B actually consists of coordinate patches with trivialisations $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$. Otherwise, we simply refine our cover so that it has this property. Assume that B has rank m . We need to define a collection, $\{V_\alpha\}$ of coordinate patches on E and maps, $\bar{\phi}_\alpha : V_\alpha \rightarrow \mathbb{R}^m \times \mathbb{R}^n$. Set $V_\alpha = [U_\alpha \times \mathbb{R}^n]$. Each V_α is open by the definition of the quotient topology. Define $\bar{\phi}_\alpha$ by $[(x, v)_\alpha] \mapsto (\phi_\alpha(x), v)$. We have the following commutative diagram:

$$\begin{array}{ccc} & V_\alpha \cap V_\beta & \\ \bar{\phi}_\beta \swarrow & & \searrow \bar{\phi}_\alpha \\ (U_\alpha \cap U_\beta) \times \mathbb{R}^n & \xleftarrow{\phi_\beta \phi_\alpha^{-1}} & (U_\alpha \cap U_\beta) \times \mathbb{R}^n \end{array}$$

On the overlap, $\bar{\phi}_\beta \bar{\phi}_\alpha^{-1}$, has the following result,

$$(x, v) \mapsto \bar{\phi}_\beta([(x, v)_\alpha]) = \bar{\phi}_\beta([(x, g_{\beta\alpha}(x)(v)]_\beta)) = (x, g_{\beta\alpha}(x)(v))$$

Thus, on the overlap, the maps are smooth since the $g_{\beta\alpha}$ are. Since we now actually have a smooth manifold, we continue. We now need to examine the projection map, $\pi : E \rightarrow B$, defined above by $[(x, v)_\alpha] \mapsto x$. Locally, we have

$$\begin{array}{ccc} & V_\alpha & \\ \pi \swarrow & & \searrow \bar{\phi}_\alpha \\ U_\alpha & \xleftarrow{\pi_1} & U_\alpha \times \mathbb{R}^n \end{array}$$

Where π_1 is the natural projection. We have that $\pi = \pi_1 \circ \bar{\phi}_\alpha$ and so π is smooth (well defined, continuous, etc). By definition,

$$\bar{\psi}_\alpha^{-1}(\pi_1^{-1}(x)) = \{[(x, v)_\alpha] : v \in \mathbb{R}^n\} \subseteq \pi^{-1}(x)$$

Do we get anything else? Suppose that $[(x, v')_\beta]$ is in the fiber over x . Then $[(x, v')_\beta] = [(x, g_{\alpha\beta}(x)(v'))_\alpha]$. So, the full fiber over x is \mathbb{R}^n . The remainder of the proof is left as the exercise below. \square

Exercise 2. Show that, in fact, the above statements also yield the locally trivial condition and the transition properties

Recall that a bundle, $\pi : P \rightarrow B$ is a principal bundle G -bundle if $\pi^{-1}(x) \cong G$, where G is a Lie group, π is locally trivial, and there is a free right G -action on P which preserves the fibers. A map of two G -bundles, P and P' is a bundle map, $h : P \rightarrow P'$ which is G -equivariant; that is, $h(p \cdot g) = h(p) \cdot g$. The locally trivial condition says that the trivialization, $\psi_\alpha : P|_{U_\alpha} \rightarrow U_\alpha \times G$ must be fiber preserving and G -equivariant; where the G -action on $U_\alpha \times G$ is trivial on U_α and group multiplication on the G factor. So, on $U_\alpha \cap U_\beta$, $\psi_\beta \psi_\alpha^{-1}$ has the form,

$$(x, g) = (x, 1g) \mapsto (\psi_\beta \psi_\alpha^{-1}(x, 1))g$$

Define $g_{\beta\alpha}(x) = (\psi_\beta \psi_\alpha^{-1}(x, 1))$. So, we must have that $(x, g) \mapsto (x, g_{\beta\alpha}(x) \cdot g)$. Exactly as before, we define

$$P = (\coprod U_\alpha \times G) / \sim$$

where \sim is defined, as before, by the $\{g_{\beta\alpha}\}$.