THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 40

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Class Notes From Math 433

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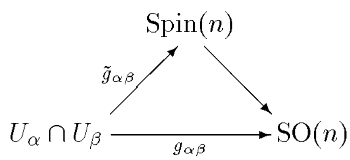
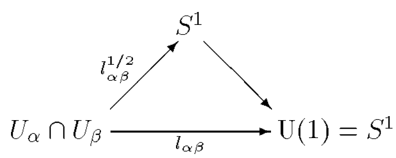
Last time we defined $\text{Spin}_C(n) = \text{Spin}(n) \times_{\mathbb{Z}_2} S^1$. $\text{Spin}_{\mathbb{C}}(n)$ was a double cover of $\text{SO}(n) \times S^1$ via $[\tilde{g}, \lambda] \mapsto (g, \lambda^2)$. Given $P_{\text{SO}(n)}$ (of the associated vector bundle E) and L (a complex line bundle with a Hermitian metric) over B with transition functions $\{g_{\alpha\beta}\}$ and $\{l_{\alpha\beta}\}$, if we $U_\alpha \cap U_\beta$ is contractible, then we can lift



and



Write $l_{\alpha\beta} = e^{i\pi x_{\alpha\beta}}$. Then,

(1) The cocycle condition on the $l_{\alpha\beta}$ says that $x_{\alpha\beta} + x_{\beta\gamma} + x_{\gamma\alpha} \in 2\mathbb{Z}$ for all α, β, γ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.
(2) $l_{\alpha\beta} = e^{i\pi x_{\alpha\beta}/2}$.

We thus get

$$l_{\alpha\beta}^{1/2} l_{\beta\gamma}^{1/2} l_{\gamma\alpha}^{1/2} = e^{i\pi w_{\alpha\beta\gamma}/2}$$

where

$$w_{\alpha\beta\gamma} = x_{\alpha\beta} + x_{\beta\gamma} + x_{\gamma\alpha}$$

Define

$$\Gamma_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} \in \mathbb{Z}_2$$

and

$$c_{\alpha\beta\gamma} = \frac{w_{\alpha\beta\gamma}}{2} \in \mathbb{Z}$$

In order for

$$[\tilde{g}_{\alpha\beta}, l_{\alpha\beta}^{1/2}] \in \text{Spin}_{\mathbb{C}}(n)$$

to define a $\text{Spin}_{\mathbb{C}}(n)$-bundle, $P_{\text{Spin}_{\mathbb{C}}}$, (e.g, a double cover $P_{\text{Spin}_{\mathbb{C}}} \to P_{\text{SO}(n)} \times L$), we need

$$[\Gamma_{\alpha\beta\gamma}, c_{\alpha\beta\gamma}] = 1\, \text{Spin}_{\mathbb{C}}(n)$$

i.e, $\Gamma_{\alpha\beta\gamma} = 1$ and $c_{\alpha\beta\gamma} \equiv 0 \bmod 2$ or $\Gamma_{\alpha\beta\gamma} = -1$ and $c_{\alpha\beta\gamma} \equiv 1 \bmod 2$.

We have that $\Gamma = \{\Gamma_{\alpha\beta\gamma}\}$ defines a Z_2 Čech cochain and $C = \{c_{\alpha\beta\gamma}\}$ defines a $\mathbb{Z}$ Čech cochain. If $\bar{c}_{\alpha\beta\gamma}$ is the $\mathbb{Z}_2$ reduction of $c_{\alpha\beta\gamma}$, then $\bar{C} = \{\bar{c}_{\alpha\beta\gamma}\}$ defines a $\mathbb{Z}_2$ Čech cochain. The condition that we want is $\Gamma + \bar{C} = 0$. Hence, these define cohomology classes and $[\Gamma] = [\bar{C}] \in H^2(B;\mathbb{Z})$.

Fact: $[\Gamma] = w_2(E)$ and $[C] \in H^2(B;\mathbb{Z})$ is the first chern class, $c_1(L)$.

We thus express the condition for the existence of $\text{Spin}_{\mathbb{C}}$ structures as $c_1(L) \equiv w_2(E) \bmod 2$. Given an oriented manifold M, if we can find a complex line bundle $L \to M$ such that $c_1(L) \equiv w_2(M) \bmod 2$, then (M, L) admits a $\text{Spin}_{\mathbb{C}}$ structure; i.e, $TM \times L$ admits a lift. If M is Spin, then we can take $L = M \times \mathbb{C}$ and use the P_{Spin} lift to construct $P_{\text{Spin}_{\mathbb{C}}}$. If $w_2(M) \neq 0$, then we may still be able to get a $\text{Spin}_{\mathbb{C}}$ structure.

Equations over Bundles

Suppose that $E \to B$ is a smooth vector bundle with a metric and structure group $\mathrm{U}(n)$. We would like to find *special* connections on E. The idea of special needs to be codified.

Given $D \in \mathcal{A} = \{$ unitary connections $\}$, we get that the curvature is an $\text{End}(E)$ valued $2-$form; i.e, $F_D \in \Omega^2(\text{End}(E))$. Given any metric on E and a metric on B (and hence on $\text{End}(E)$), we can measure the length of F_D, $|F_D|$.

We can define the *Yang-Mills* functions,

$$YM(D) = \int_B |F_D| \text{ dvol} = ||F_D||^2_{l^2}$$

This defines a map, $YM : \mathcal{A} \to \mathbb{R}$.

The special connections on E are the critical points of YM. In certain special cases, e.g, when B is a $4-$manifold and E is rank 2, you can identify a condition for the minimizers of YM. This is expressed as a certain set of partial differential equations called the Anit-Self Dual Equations (ASD). The solutions to the ASD equations are called *instantons*.

Facts: The set, {instantons} is a moduli space (a nice geometric object). This can be used to define algebraic-topological invariants of B.

Given a $4-$manifold M, erect $E \to M$ with metrics. Take $\mathcal{A}$ to be the connections as before and look at the moduli space of instantons. This defines algebraic-topological invariants of M.

Alternatively, we can take a metric on M and look at $\Omega^p(M)$. Consider Laplaces equation, $dd^* = d^*d = 0$. The space of solutions consists of $p-$harmonic forms, $\mathcal{H}^p$. The dimension, $\dim\mathcal{H}^p = b_p$ is an interesting invariant.

The new version of the two above construction goes like this. Start with a line bundle $L \to M$ with $c_1(L) \equiv w_2(M) \bmod 2$. We get a $\text{Spin}_{\mathbb{C}}$ structure associated to (M, L). There are two distinguished one, S^+ and S^-, both rank 2 spinor bundles. We get the ASD equations for a connection D_L on L and a section $\Phi \in \Omega^0(S^+)$. Solutions to these yield more invariants.

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