THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 5

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Recall that we can describe a vector bundle, $\pi : E \to B$, in terms of local trivializations over a cover of B together with transition functions, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$, where n is the ranks of the bundle. We had that

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

Similarly, for principal bundles,

$$P = (\coprod U_\alpha \times G)/\{g_{\alpha\beta}\}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ and the relation was $(b, v)_\alpha \sim (b, g_{\beta\alpha}(b) \cdot v)_\beta$.

An important observation can be made here. Given a vector bundle, $\pi : E \to B$ given by

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})\}$$

we can build a principal $\mathrm{GL}(n, \mathbb{R})-$bundle. Set

$$P = (\coprod U_\alpha \times \mathrm{GL}(n, \mathbb{R}))/\{g_{\alpha\beta}\}$$

We say that E is associated to P. A natural question to ask is if the converse holds. That is, given a principal $G-$bundle, $\pi : P \to B$, can we build an associated vector bundle? To build such a bundle, we need a group representation, $\rho : G \to \mathrm{GL}(V)$ for some vector space V.

Example 1. If $G = \mathrm{GL}(n, \mathbb{R})$, then just use the standard representation on $\mathbb{R}^n$. If $G = \mathrm{O}(n)$, then we may embed $\mathrm{O}(n)$ in $\mathrm{GL}(n, \mathbb{R})$ and use the composite representation. If $G = \mathrm{U}(n)$, the unitary group, we get, via the embedding of $\mathrm{U}(n)$ into $\mathrm{GL}(n, \mathbb{C})$, a representation on $\mathbb{C}^n$. We can get a real representation by thinking of $\mathrm{GL}(n, \mathbb{C})$ as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$.

We define

$$E = (\coprod U_\alpha \times V)/\{\rho(g_{\alpha\beta})\},$$

where

$$\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$$

The cocycle condition is easily verfied, for,

$$\rho(g_{\alpha\beta})\rho(g_{\beta\gamma})\rho(g_{\gamma\alpha}) = \rho(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}) = \rho(\mathrm{Id}) = \mathrm{Id}$$

Thus far, we have concentrated on local descriptions of associated bundles. It is also useful to have a global description of such things. To do so, we need the following defintion.

Definition 1. Given $\pi : P \to B$ a principal $G-$bundle and a representation $\rho : G \to \mathrm{GL}(V)$, we define the *product of P and V over G* by

$$E = P \times_G V,$$

where E is a quotient of $P \times V$ under the relation $(p, v) \sim (p \cdot g, \rho(g^{-1})v)$ for all $g \in G$.

Claim. *Under $[(p,v)] \mapsto \pi(p)$, we get a vector bundle, $E \to B$ with fiber V. Furthermore, if*

$$P = (\coprod U_\alpha \times G)/\{g_{\alpha\beta}\}$$

then

$$E = (\coprod U_\alpha \times V)/\{\rho(g_{\alpha\beta})\}$$

That is, the two constructions are the same.

Exercise 1. Prove the above claim. Hint: We can identify

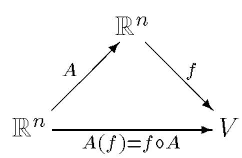
$$(U_\alpha \times G) \times_G V \to U_\alpha \times V$$

by $[((b,u)_\alpha, v)] \mapsto (b, \rho(u)v)$ with inverse map $(b,v) \mapsto [((b,e),v)]$, where $e \in G$ is the group identity.

We now give a geometric interpretation of the principal $\mathrm{GL}(n,\mathbb{R})$ bundle associated to a rank n vector bundle $\pi : E \to B$.

Definition 2. Given a real vector space V of dimension n, a frame for V is an identification of $\mathbb{R}^n$ with V, $f : \mathbb{R}^n \to V$. This is equivalent to choosing a basis; that is, if $\{e_i\}$ is the standard basis for $\mathbb{R}^n$, then $\{f(e_i)\}$ is the basis we have chosen for V. Define $F(V)$ to be the set of all frames of V.

Observe that $\mathrm{GL}(n,\mathbb{R})$ acts on $F(V)$ by:



Exercise 2. Show that $F(V) \cong \mathrm{GL}(n,\mathbb{R})$.

For a vector bundle, $\pi : E \to B$ and for each $b \in B$, we have that $E_{|b} \cong \mathrm{GL}(n,\mathbb{R})$. So, we get a copy of $\mathrm{GL}(n,\mathbb{R})$ attached to each b via $F(E_{|b}) = P_b$.

Claim. *If*

$$\bigcup_{b \in B} P_b = P$$

then P is a principal $GL(n,\mathbb{R})-$bundle. This is the same as the previous construction. This is called the **principal frame bundle**, *or the* **principal bundle of frames**.

Proof: The proof is left as an exercise, with the following hint. Say

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

with trivializing maps, $\psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^n$. Use $\psi_\alpha^{-1}(b)() : \mathbb{R}^n \to E_b$ to define a frame, f. We then get a description of P_b as

$$P_b = \{f_\alpha \circ A : A \in \mathrm{GL}(n,\mathbb{R})\},$$

and hence, $P|U_\alpha \cong U_\alpha \times \mathrm{GL}(n,\mathbb{R})$ via the map $(b,A) \mapsto f_\alpha(b)A$. Notice that the P_b carries a right action of $\mathrm{GL}(n,\mathbb{R})$.

Exercise 3. Show that P also carries a right action.

Exercise 4. Check the transition function conditions.

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