THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 6

Professor Steven Bradlow
Class Notes From Math 433

University of Illinois at Urbana-Champaign

February 4, 1998

Recall that given a rank n vector bundle, we were able to construct a principal $\mathrm{GL}(n, \mathbb{R})$ bundle. Furthermore, given a principal $G-$bundle, P, and a representation $\rho : G \to \mathrm{GL}(V)$, we constructed a vector bundle, $E = P \times_G V = P \times_\rho V$. Furthermore, if V is of rank n, then

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

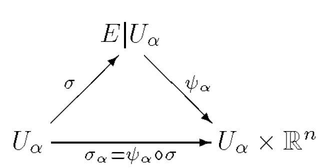
where the $g_{\alpha\beta} : U_\alpha \to U_\beta \to \rho(G) \subseteq \mathrm{GL}(n, \mathbb{R})$ are the transition functions. In general, we say that the structure group of a bundle can be reduced if its structure group G is a subset of $\mathrm{GL}(n, \mathbb{R})$. In general, a vector bundle can be reduced from $\mathrm{GL}(n, \mathbb{R})$ to some subgroup, G, if and only if we can find a system of local trivializations such that all transition functions take their values in G.

Local Description of a Section

Suppose that a vector bundle,$\pi : E \to B$, is described as

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

and that $\sigma : B \to E$ is a section. Let $\psi_\alpha : E_{|U_\alpha} \to U_\alpha \times \mathbb{R}^n$ be the trivializing maps. We had that the following diagram was commutative:

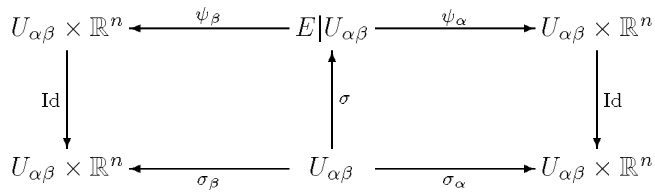


Use the fiber preserving properties of the maps involved to check that $\sigma_\alpha(b) = (b, s_\alpha(b))$ where $s_\alpha : U_\alpha \to \mathbb{R}^n$. Now, σ determines a collection of local sections,

$$\{\sigma : U_\alpha \to \mathbb{R}^n\}$$

one for each trivializing open set.

The natural question to ask here is if given a collection of local sections, $\{\sigma_\alpha : U_\alpha \to \mathbb{R}^n\}$, one for each trivializing open set, when does the collection determine a global section, $\sigma : B \to E$ with $\sigma_{|U_\alpha} = \sigma_\alpha$? On $U_\alpha \cap U_\beta = U_{\alpha\beta}$, we have the following commutative diagram:



Writing $g_{\beta\alpha} = \psi_\beta \psi_\alpha^{-1}$, we see that $s_\beta = g_{\beta\alpha} s_\alpha$. Hence, given a collection of local sections, $\{\sigma_\alpha(b) = (b, s_\alpha(b))\}$ such that $s_\beta(b) = g_{\beta\alpha}(b) s_\alpha(b)$, then they define a global section by defining $\sigma(b) = \psi_\alpha^{-1} \circ \sigma_\alpha(b)$.

Example 1. Set $s_\alpha(b) = 0$ for all $b \in U_\alpha$. This defines the local zero section and the collection $\{s_\alpha\}$ will glue together to define the global zero section.

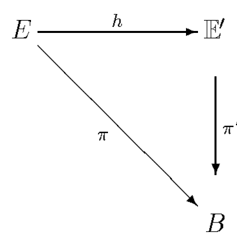
Observe that the same analysis of sections for vector bundles applies to sections of principal $G-$bundles. In other words, given a principal $G-$bundle, described by

$$P = (\coprod U_\alpha \times G)/\{g_{\alpha\beta}\},$$

the description of a section is the same as above, but with $\mathbb{R}^n$ replaced by G. Now, in the relation, $s_\beta = g_{\beta\alpha} s_\alpha$, the multiplication on the right hand side is group multiplication. Conversely, using a section, say σ, the transition functions can be written as $g_{\beta\alpha} = s_\beta s_\alpha^{-1}$. The significance of this will become clear shortly.

A Return to Bundle Maps

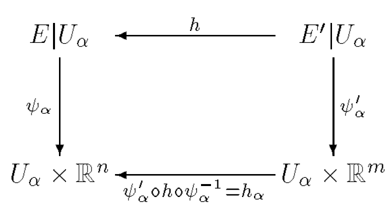
Suppose that we have a bundle map,



with

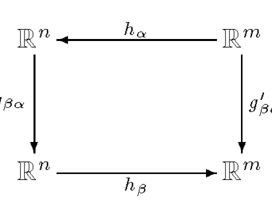
$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$
$$E' = (\coprod U_\alpha \times \mathbb{R}^m)/\{g'_{\alpha\beta}\}$$

Then, we have the following commutative diagram specified by the trivializations for E and E' respectively,



Exercise 1. Show that we can regard $h_\alpha : U_\alpha \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. That is, we can think of h as an element of $\mathrm{Mat}_{n,m}(\mathbb{R})$, real n by m matrices.

The exercise says that given h, we obtain a set, $\{h_\alpha : U_\alpha \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)\}$. How are h_α and h_β related? Using the transition functions from E and E' respectively, we require that the following diagram commute:



Consider the special case of $E = E'$. Then h is a bundle endomorphism and

$$h_\alpha : U_\alpha \longrightarrow \mathrm{End}(\mathbb{R}^n, \mathbb{R}^n) \cong \mathrm{Mat}_{n,n}(\mathbb{R})$$

are such that $h_\beta g_{\beta\alpha} = g_{\beta\alpha} h_\alpha$. If h is, in fact, a bundle automorphism then,

$$h_\beta = g_{\beta\alpha} h_\alpha g_{\beta\alpha}^{-1} = g_{\beta\alpha} h_\alpha g_{\alpha\beta}$$

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu