THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 7

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Class Notes From Math 433

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Recall from last time that if E and E' are bundles and $h : E \to E'$ is a bundle map, then we form a collection of local maps, $\{h_\alpha : U_\alpha \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)\}$ such that $g'_{\beta\alpha} h_\alpha = h_\beta g_{\beta\alpha}$. If h was, in fact, an automorphism, then we had $h_\beta = g_{\beta\alpha} h_\alpha g_{\alpha\beta}$.

It is important to know when a bundle, E, over B is isomorphic to the trivial bundle, $E \cong B \times \mathbb{R}^n$. Suppose E is given by

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

Observe that the trivial bundle, $\underline{\mathbb{R}^n}$, is given by

$$\underline{\mathbb{R}^n} = B \times \mathbb{R}^n = (\coprod U_\alpha \times \mathbb{R}^n)/\{g'_{\alpha\beta}\}$$

where $g'_{\alpha\beta}$ is identically the identity. A map $h : E \to \underline{\mathbb{R}^n}$is thus described by a collection of locally defined maps, $\{h_\alpha : U_\alpha \to \text{GL}(n, \mathbb{R})\}$ satisfying $g_{\beta\alpha} = h_\beta h_\alpha^{-1}$. That is, if we can find a collection of such h_α, then we can trivialize the bundle.

But, this says precisely that the associated principal $\text{GL}(n, \mathbb{R})-$bundle is trivial if and only if it admits a section. It follows in the same way that any principal bundle is trivial if and only if it admits a section. So, unlike the case for vector bundles, sections of principal bundles completely detect whether or not a bundle is trivial.

Side bar. For those who know some homotopy. The zero section, $s : B \to E$ embeds B diffeomorphically into E. Furthermore, this diffeomorphic copy of B in E is a deformation retract of E. Hence, B and E have the same homotopy type. So, to distinguish bundles, homotopy and homology theory will be difficult to use.

New Bundles From Old Bundles

Constructions done on vector spaces will carry over almost directly to the case for vector bundles. For example, given two vector spaces, V_1, V_2, we can form new vector spaces, $V_1 \oplus V_2, V_1 \otimes V_2, V_1^*, V_1 \wedge V_1$,etc. These all have bundle versions. If E_1 is a rank r_1 bundle and E_2 is a rank r_2 bundle, then we construct a new bundle, $E_1 \oplus E_2$ with rank $r_1 + r_2$ as follows: Suppose that E and E' are represented by

$$E_1 = (\coprod U_\alpha \times \mathbb{R}^{n_1})/\{g^1_{\alpha\beta}\}$$
$$E_2 = (\coprod U_\alpha \times \mathbb{R}^{n_2})/\{g^2_{\alpha\beta}\}$$

We define the sum of the bundles to be

$$E_1 \oplus E_2 = (\coprod U_\alpha \times \mathbb{R}^{n_1+n_2})/\{g^{\oplus}_{\alpha\beta}\}$$

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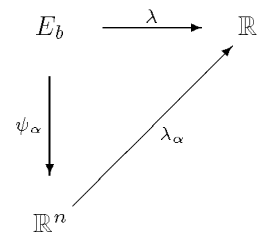
where $g^{\oplus}_{\alpha\beta}$ is the n_1 by n_2 matrix with $g^1_{\alpha\beta}$ in the upper left corner and $g^2_{\alpha\beta}$ in the lower right corner. To check that does define a bundle, we need to check the cocycle condition. This does, however, check out just by using that the two blocks of the matrix satisfy it. $E_1 \otimes E_2$ is constructed similarly.

The construction of the dual bundle, E^*, is slightly more difficult. We want the fibers of the new bundle to be $E^*_b \cong \mathrm{Hom}(E_b, \mathbb{R})$. If E is given by

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

then we really need to concentrate on what the $g^*_{\alpha\beta}$ should be. Set $g^*_{\alpha\beta} = (g^t_{\alpha\beta})^{-1}$, the transpose-inverse of the original transition functions. The $g^*_{\alpha\beta}$ will satisfy the cocycle condition and so will define a bundle. But, is it the correct bundle? That is, do we have $E^*_b \cong \mathrm{Hom}(E_b, \mathbb{R}^n)$? Suppose that $\psi_\alpha : E_b \to \mathbb{R}^n$ is an identification of the fiber and $\lambda \in \mathrm{Hom}(E_b, \mathbb{R})$. Then, we get a map $\lambda_\alpha : \mathbb{R}^n \to \mathbb{R}$. This is expressed in the following diagram:



We get a correspondance, $\lambda \leftrightarrow \lambda_\alpha$ and so we get $E^*_b \cong \mathbb{R}^n \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$. In fact, we also get $E|U_\alpha \cong U_\alpha \times \mathbb{R}^n$. Over $U_\alpha \cap U_\beta$, we have $\lambda_\beta = \lambda \circ \psi_\beta^{-1}$ and $\lambda_\alpha = \lambda \circ \psi_\alpha^{-1}$. So,

$$\lambda = \lambda_\beta \psi_\beta = \lambda_\alpha \psi_\alpha = \lambda_\alpha \circ (\psi_\alpha \circ \psi_\beta^{-1}) \circ \psi_\beta = \lambda_\alpha g_{\beta\alpha} \psi_\beta$$

The compatibility condition on the $\{\lambda_\alpha\}$ is that, as maps $\mathbb{R}^n \to \mathbb{R}$, they satisfy $\lambda_\beta = \lambda_\alpha \circ g_{\alpha\beta}$.

Exercise 1. Show that this is equivalent to the requirement that

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{(g^t_{\alpha\beta})^{-1}\}$$

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