THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 8

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Recall from last time that given

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\},$$

we can use $\{(g^t_{\alpha\beta})^{-1}\}$ to define $E^* = (\coprod U_\alpha \times \mathbb{R}^n)/\{(g^t_{\alpha\beta})^{-1}\}$. Then $E^*_b = \mathrm{Hom}(E_b, \mathbb{R})$. All we need: If a vector space V has a basis $\{v_1, \dots, v_n\}$, then we can define the dual basis $\{v^*_1, \dots, v^*_n\}$ for $V^* = \mathrm{Hom}(V, \mathbb{R})$, where $v^*_i : V \to \mathbb{R}$ defined by

$$v^*_i(v_j) = \delta_{ij}.$$

Then any $\lambda \in V^*$ is given by $\sum_i \lambda_i v^*_i$.

For complex bundles:

$$E = (\coprod U_\alpha \times \mathbb{C}^n)/\{g_{\alpha\beta}\}$$

and

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C}).$$

Exactly the same proof shows that $\{{g^t_{\alpha\beta}}^{-1}\}$ defines E^* and we can identify $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}) \cong \mathbb{C}^n$ in the same way and see that $E^*_b = \mathrm{Hom}(E_b, \mathbb{C})$.

Remarks. There are some points to note:

(1) We can use a Hermitian metric to identify

$$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}) \xrightarrow{\simeq} \mathbb{C}^n,$$

but this is NOT a $\mathbb{C}$-linear map, so it is NOT the identification we want!

(2) Real case: Suppose that we can pick a local trivialization $E|_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times \mathbb{R}^n$ such that

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R}).$$

(so structure groups of E can be reduced to $\mathrm{O}(n)$) Then

$$g^t_{\alpha\beta} g_{\alpha\beta} = I.$$

So

$${g^t_{\alpha\beta}}^{-1} = g_{\alpha\beta}.$$

Conclusion:

$$E^* \cong E.$$

(3) Complex case: Even if

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C}).$$

Then

$$\overline{g_{\alpha\beta}}^t g_{\alpha\beta} = I.$$

So

$$g_{\alpha\beta}^{t\ -1} = \overline{g_{\alpha\beta}} \neq g_{\alpha\beta}.$$

Therefore $E^* \ncong E$ as complex bundles. In fact $E^* \cong \overline{E}$.

Metrics on Bundles

Suppose that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(n)$ for $E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$.

Claim. *We can define a fiberwise inner product on each E_b.*

Proof: Define $\langle \cdot , \cdot \rangle_b$ on E_b, for each $b \in B$ as follows: Let $(\cdot , \cdot)$ be the standard inner product on $\mathbb{R}^n$, that is, $(e_i, e_j) = \delta_{ij}$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. Then we can identify $E_b \overset{\Psi_\alpha}{\xrightarrow{\simeq}} \mathbb{R}^n$ for any α such that $b \in U_\alpha$ and define $e_i^\alpha(b) = \Psi_\alpha^{-1}(b, e_i)$.

Exercise 1. Show that $\{e_i^\alpha(b)\}$ is a basis for E_b.

Now we define $\langle \cdot , \cdot \rangle_b$ by declairing $\{e_i^\alpha(b)\}$ to be an orthonormal frame, that is, by defining

$$\langle e_i^\alpha(b), e_j^\alpha(b) \rangle_b = \delta_{ij}.$$

Thus if $v = \Sigma v_i e_i^\alpha(b)$ and $u = \Sigma u_i e_i^\alpha(b)$, then

$$\begin{aligned} \langle u, v \rangle_b &= (\Psi_\alpha(b, v_1), \Psi_\alpha(b, v_2)) \\ &= \sum_{i=1}^n u_i v_i. \end{aligned}$$

□

Exercise 2. Show that this is well-defined if $g_{\alpha\beta} \in \mathrm{O}(n)$.

Exercise 3. Show that if σ_1 and σ_2 are C^∞ sections, then $b \mapsto \langle \sigma_1(b), \sigma_2(b) \rangle_b$ is a smooth map.

Exercise 4. Suppose that $\Psi : E|_U \xrightarrow{\simeq} U \times \mathbb{R}^n$ is a local trivialization and that we define $\{f_i(b) = \Psi^{-1}(b, e_i)\}_{i=1}^n$. (This is a basis for E_b for every $b \in U$ and $\{f_i\}$ is called a **local frame**.)

Define

$$H_{ij}(b) = \langle f_j(b), f_i(b) \rangle_b,$$

show that the map $b \mapsto H_{ij}(b)$ defines a smooth map from U to $\mathrm{GL}(n)$.

Remark. An assignment of an inner product $\langle \cdot , \cdot \rangle_b$ to each fiber of E is called a **smooth bundle metric** if either (hence both!) of the maps in Exercises 3 and 4 are smooth.

Exercise 5. Suppose E admits a smooth bundle metric, that is, $\langle \cdot , \cdot \rangle_b$ on each E_b, smoothly varying with b. Show that we can pick $E|_U \xrightarrow{\simeq} U \times \mathbb{R}^n$ such that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(n)$.

Examples of existence of metrics.

1. On

$$\begin{array}{c} B \times \mathbb{R}^n \\ \big\downarrow \\ \mathbb{R}^n \end{array}$$

Take $\langle \cdot , \cdot \rangle_b = (\cdot , \cdot)$ (or any other metric on $\mathbb{R}^n$.)

2. On $E = \coprod U_\alpha \times \mathbb{R}^n$, take a partition of unity subordinate to $\{U_\alpha\}$—Need $\{U_\alpha\}$ locally finite here! That is, $\{\rho_\alpha : U_\alpha \to \mathbb{R}$ is smooth$\}$ such that $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$ and $\Sigma_\alpha \rho_\alpha(b) = 1, \forall\, b$. At $b \in U_\alpha$, we can define $\langle \cdot\,, \cdot \rangle_{b,\alpha}$ using $\Psi_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$. Then we can set $\langle \cdot\,, \cdot \rangle_b = \Sigma_\alpha \rho_\alpha(b) \langle \cdot\,, \cdot \rangle_{b,\alpha}$.

Exercise 6. Show that this definition defines an inner product.

Note. Similarly for Hermitian metrics on complex vector bundles. Replacing "orthonormal" by "unitary" and $\mathrm{O}(n)$ by $\mathrm{U}(n)$, all the expeted analogs of the above results apply.

Exercise 7. Fill in the details.

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