THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 12

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Class Notes From Math 433

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We will need to be familiar with Grassmanians and bundles over them for our construction of the unviversal bundle.

Definition 1. The real *Grassmanian* is the set of all $k-$planes in $\mathbb{R}^n$ and is denoted by $G_{\mathbb{R}}(k,n)$. Similarly, the complex *Grassmanian*, $G_{\mathbb{C}}(k,n)$ is the set of all complex $k-$planes in $\mathbb{C}^n$.

Example 1. $G(1,n)$ is the set of all lines in $\mathbb{R}^n$. This should be familiar as real projective $n-1$ space, $\mathbb{RP}^{n-1}$. Similarly, $G_{\mathbb{C}}(1,n) = \mathbb{CP}^{n-1}$.

We need to establish the following key properties of the Grassmanian. We derive the properties for $G(k,n) = G_{\mathbb{R}}(k,n)$ but the same hold for $G_{\mathbb{C}}(k,n)$. These are:

(a) $G(k,n)$ is a smooth manifold of dimension $k(n-k)$.

(b) Let γ_k^n be the subset of $G(k,n)\times\mathbb{R}^n$ consisting of $\{(V,\vec{v}) : \vec{v}\in V\}$. Then under the natural projection map, $\pi : \gamma_k^n \to G(k,n)$ is a bundle with fiber a $k-$plane.

Recall that the trivial bundle is denoted by $\underline{\mathbb{R}^n}$. Then the universal quotient bundle, Q, over $G(k,n)$ is defined by the exact sequence
$$0 \to \gamma_k^n \to \underline{\mathbb{R}^n} \to Q \to 0$$
We have that $Q_{|V} \cong \mathbb{R}^n/V$, an $n-k$-plane.

We begin with the first property above. A useful was to specify a frame is by tge $n\times k$ matrix, $[\vec{v}_1,\dots,\vec{v}_k]$ (where eacg $\vec{v}_i$ is displayed as an $n\times 1$ column). The linear independence condition is equivalent to the condition that $[\vec{v}_1,\dots,\vec{v}_k]$ has rank k. We can interpret this matrix as an injective linear map, $\mathbb{R}^k \to \mathbb{R}^n$.

Of course each $k-$plane admits may frames, so that the description is in no way unique. In fact, given any $A\in \mathrm{GL}(k)$, we get a new frame described by the matrix, $[\vec{v}_1,\dots,\vec{v}_k]\cdot A$, or equivalently, the map
$$\mathbb{R}^k \xrightarrow{A} \mathbb{R}^k \xrightarrow{[\vec{v}_1,\dots,\vec{v}_k]} \mathbb{R}^n$$
This leads to a description of $G(k,n)$ as a quotient (by $\mathrm{GL}(k)$) of the space of all *k-frames*. The latter is called the Stiefel manifold of $k-$frames and is denoted by $F(k,n)$. We can identify
$$\begin{aligned} F(k,n) &= \{n\times k \text{ matrices of rank} k\} \\ &= \{f:\mathbb{R}^k\to\mathbb{R}^n : f \text{ is linear and injective}\}\end{aligned}$$

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Then, $G(k,n) = F(k,n)/\operatorname{GL}(n)$ where $\operatorname{GL}(k)$ acts on $F(k,n)$ in the manner described above.

Exercise 1. Show that the $\operatorname{GL}(k)$ action on $F(k,n)$ is a free right action.

Exercise 2. Show that $F(k,n)$ is a smooth manifold of dimension kn.

Suppose that $[\vec{v}_1, \ldots, \vec{v}_k] \in G(k,n)$ has the property that the first k by k minor has non-zero determinant (since the matrix has rank k, at least one of the minors has non-zero determinant). Then, there is some $A \in \operatorname{GL}(k)$ such that

$$[\vec{v}_1, \ldots, \vec{v}_k] \cdot A = [\frac{\mathrm{Id}}{B}]$$

That is, the first k rows have been reduced to the identity, and B is an $(n-k)$ by k block. We define a map,

$$\operatorname{Mat}(n-k,k) \to U_{1,\ldots,k} \subset \operatorname{GL}(k,n)$$

by

$$B \mapsto [\frac{\mathrm{Id}}{B}]$$

Exercise 3. If $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, then U_I defines a coordinate patch for $G(k,n)$. Show that $G(k,n)$ is covered by the set of all such U_I and that the coordinate transformations are smooth.

Note that the story gets better. Over $\mathbb{C}^n$, the coordinate transformations for $G_{\mathbb{C}}(k,n)$ are holomorphic. Thus, $G_{\mathbb{C}}(k,n)$ is a complex manifold of dimension $k(n-k)$.

We now move to the second property. We first look at the projection $\pi : F(k,n) \to G(k,n)$. This clearly has fiber $\operatorname{GL}(k)$. In fact, it is a $\operatorname{GL}(k)$ bundle. We need to establish the local triviality over U_I. Over $U_{\{1,\ldots,k\}}$, describe

$$[\vec{v}_1, \ldots, \vec{v}_k] = [\frac{\mathrm{Id}}{B}] \cdot A$$

Define a map $[\vec{v}_1, \ldots \vec{v}_k] \mapsto ([\vec{v}_1, \ldots, \vec{v}_k], A)$. This is the trivializing map. So, $\pi : F(k,n) \to G(k,n)$ is a principal $\operatorname{GL}(k)$-bundle. How does this bundle relate to γ_k^n?

Claim.

$$\gamma_k^n = F(k,n) \times_{GL(k)} \mathbb{R}^k$$

Exercise 4. Establish this for the case where $k = 1$. the tautological bundle over $\mathbb{RP}^{n-1}$.

Remark. Given a $k-$plane, $V^k \subseteq \mathbb{R}^n$, we obtain an $n-k$ plane given by $\mathbb{R}^n/V^k$ (if we have a metric, then we can identify this with $V^\perp$m the othogonal complement of V).

We thus get a map $q : G(k,n) \to G(n-k,n)$. We can use this map to pull back $Q \to G(n-k,n)$ to $G(k,n)$. We then have the following diagram:

$$\begin{tikzcd}
\gamma_k^n \arrow[dr] & q^*(Q) \arrow[d] & Q \arrow[d] \\
 & G(k,n) \arrow[r, "q"'] & G(n-k,n)
\end{tikzcd}$$

A natural question to ask is: how are γ_k^n and $q^*(Q)$ realted. It turns out that γ_k^n is the same as the dual bundle to $q^*(Q)$.

Exercise 5. Prove that $\gamma_k^n \cong (q^*(Q))^*$.

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