THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 13

Professor Steven Bradlow
Class Notes From Math 433

University of Illinois at Urbana-Champaign

February 20, 1998

Last time we constructed a bundle, $\pi : Q \to G(N-n, N)$ with $Q_{|V} \cong \mathbb{R}^n/V$. We saw that for a rank n bundle, $E \to B$ over a compact space with a finite good cover, that there is a collection of sections, $\{s_1, \dots, s_N\}$ which have the property that the evaluation map, $\mathrm{ev}_b : \mathbb{R}^n \to E_b$ (via $e_i \mapsto s_i(b)$), generates the fiber. The kernel of ev_b fits into an exact sequence,

$$0 \to \ker \mathrm{ev}_b \to \mathbb{R}^N \to E_b \to 0$$

Hence, we can construct a map $f : B \to G(N-n, N)$ by $b \mapsto \ker \mathrm{ev}_b$.

Claim. *$f^*(Q) \cong E$.*

Proof:

$$F^*(Q)_{|b} \cong Q_{|(b)} \cong [\ker \mathrm{ev}_b] \cong \mathbb{R}^N/\ker \mathrm{ev}_b \cong E_b$$

□

Recall that this construction is meant to produce a map, $\mathrm{Vect}_n(B) \to [B, BG]$ which is the inverse of the map, $[f] \mapsto [f^*(Q)]$. We thus need to consider what happens if we pick a different set of sections, $\{s'_1, \dots, s'_n\}$ and thus get a different map, $f' : B \to G(N-n, N)$. We would like to conclude that f and f' are homotopic, i.e,

$$[f] = [f'] \quad \text{in } [B, BG]$$

In order for this to be true, we need to embed

$$G(N-n, N) \to G(N'-n, N')$$

for $N' > N$. This embedding is induced naturally by the embedding

$$\mathbb{R}^N \to \mathbb{R}^{N'}$$

in which $\mathbb{R}^N$ occupies the subspace spanned by the first N standard basis vectors in $\mathbb{R}^{N'}$. The extra room in $G(N'-n, N')$ is needed in order to construct the required homotopy between the maps f and f'.[Details can be found in *Husemoller* or *Milnor* and *Stashoff*].

We can avoid questions about which N' to use in $G(N'-n, N')$ by taking the limit,

$$\dots \to G(N-n, N) \to G(N+1-n, N+1) \to \dots$$

That is, by using

$$G_\infty(n) = \lim_{N\to\infty} G(N-n, N)$$

We can similarly take the limit,

$$\dots \to \gamma_n^k \to \gamma_{n+1}^k \to \dots$$

to produce a bundle, $\gamma^k \to G_\infty(n)$ with $BG = G_\infty$ and $EG = Q$. We thus get:

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Proposition. *There is a bijective correspondance,*

$$Vect_n(B) \leftrightarrow [B, BG]$$

Remark. The above methods can be extended to the case where B is only *paracompact.*

Example 1. We compute $\mathrm{Vect}_n(S^1)$. We need to examine $G(N-n, N)$ where N is the number of elements in a good cover of S^1. It is straightforward to show that any good cover of S^1 must involve at least 3 elements. Set $N = 3n$. Now, $G(2,3) \approx G(1,3) \approx \mathbb{R}P^2$. The work above actually shows that $\mathrm{Vect}_n[S^1, \mathbb{R}P^2] = \mathbb{Z}_2$. Note that $[,]$ refers to **unbased** homotopies.

For any principal $G-$bundle, there is a construction for a spaces BG, EG and a principal $G-$bundle, $EG \to BG$. If we define $\mathrm{Prin}_G(B)$ to be the set of isomorphism classes of principal $G-$bundles over B, then we get $\mathrm{Prin}_G(B) \cong [B, BG]$. EG is obtained by something called the Milnor construction (pg 54 of Husemoller). EG is obtained by the infinite join of the group G. This is by definition

$$G * G * \cdots = \{(t_1 x_0, t_1 x_1, \dots) : t_i \in [0,1], x_i \in G, \text{ a finite number of the } t_1 \neq 0, \sum t_i = 1\}$$

Now, let G act on EG by right multiplication. Let BG be the orbit space and $EG \to BG$ the natural map. It remains to be shown that this is a principal $G-$bundle with the appropriate universal properties.

Note. EG, I believe, is the realization of the Bar construction for the space G.

Exercise 1. Take $G = \mathbb{Z}_2$. Show that EG is infact S^∞ (the limit of the system $\dots S^n \to S^{n+1} \to \dots$) and that $BG = \mathbb{RP}^\infty$ (the limit of the system $\cdots \to \mathbb{RP}^n \to \mathbb{RP}^{n+1} \to \dots$). If $G = S^1$, show that $EG = S^\infty$ and that $BG = \mathbb{CP}^\infty$ (the limit of the system $\cdots \to \mathbb{CP} \to \mathbb{CP}^{n+1} \to \dots$)

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu