THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 19

Professor Steven Bradlow
Class Notes From Math 433

University of Illinois at Urbana-Champaign

March 6, 1998

Recall from last time that given a parallel section, $s \in \Omega^0(B, E)$, we have that
(i) $s(B)$ is a submanifold of E (actually true for any section).
(ii) $T_{s(b)}(s(B)) = s_*(T_bB) = H_{s(b)} \subset T_{s(b)}E$. Given a parallel frame, $\{s_1, \dots, s_k\}$, where k is the rank of the bundle, we get such a submanifold through every point. Call $s(B)$ an integrable submanifold for $H_{(s(b)}$.

Conclusion: If we have parallel frames, then the horizontal distribution,

$$\cup_{e \in E} H_e = H \subseteq TE$$

is integrable.

Remark. Conversely, if H is integrable, then there exists an integrable submanifold through every point. This implies that if $\{e_i(b)\}$ is a basis for E_b, then we can extend this to a local parallel frame.

Exercise 1. Supply the details to prove this remark.

Obstructions to Integrability

[The following material relies upon *Frobenius's* Theorem. For a complete discussion of this, see *Warner's* book.] Suppose that we have a distribution, $\mathcal{D} \subset TM$, with the dimension of $\mathcal{D}_x = d$ for all x, e.g, $\mathcal{D}_x \subseteq T_xM$ is a dimension d subspace. When is $\mathcal{D}$ an integrable distribution?

By Frobenius' Theorem, a condition for being integrable is that for any $X, Y \in \mathcal{D}$, the Lie bracket of X and Y, $[X, Y]$, is also in $\mathcal{D}$. Recall that the Lie bracket is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

We call $\mathcal{D}$ involutive if this applies.

There is (for our purposes) a more convenient version of this condition, given in terms of differential forms. Consider the ideal in $\wedge^* T^*M$ which annihilates $\mathcal{D}$: Set

$$I_{\mathcal{D}} = \{\alpha \in \wedge^1 T^*M : \alpha(X) = 0 \text{ for all } X \in \mathcal{D}\}$$

and

$$\mathcal{I}_{\mathcal{D}} = \{\beta \wedge \alpha : \alpha \in I_{\mathcal{D}}\}$$

Example 1. Suppose $M = \mathbb{R}^n$ and

$$\mathcal{D}_x = \mathbb{R}\{\frac{\partial}{\partial x_1}\} \subseteq T_x M$$

Then $\mathcal{I}_{\mathcal{D}}$ is generated by $\{dx_1, \ldots, dx_n\}$.

The condition on $\mathcal{I}_{\mathcal{D}}$ corresponding to $\mathcal{D}$ being involutive is $d\mathcal{I}_{\mathcal{D}} \subseteq \mathcal{I}_{\mathcal{D}}$.
That is, $\mathcal{I}_{\mathcal{D}}$ is a differential ideal.

In fact, Frobenius' Theorem says that the following are equivalent
(a) $d\mathcal{I}_{\mathcal{D}} \subseteq \mathcal{I}_{\mathcal{D}}$
(b) $\mathcal{D}$ is involuted
(c) For any $x \in M$, there exists an integrable submanifold passing through x, e.g if $i : S \to M$, then $i_*(T_s S) = \mathcal{D}_x$. This is illustrated in figure 19.1

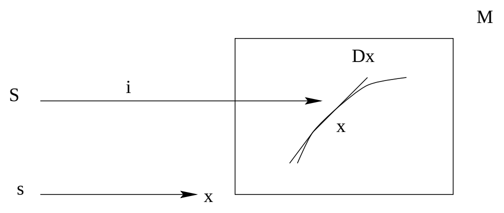

Figure 19.1

In order to apply this to $H \subset TE$, we need to calculate $\mathcal{I}_H$.

Claim. *$\mathcal{I}_H$ is generated by $dy_i + A_{ij} y_j$.*

Explaination: Pick a local frame, $\{e_i\}$ and with respect to this local frame, the connection is $D = d + A$, as before. Pick local coordinates $(x_1, \ldots, x_n)$ on B and on fibers $(y_1, \ldots, y_k)$. Define a map $\mathbb{R}^n \times \mathbb{R}^k \to E$ by

$$(x_1, \ldots, x_n, y_1, \ldots, y_k) \mapsto \sum y_i e_i(x_i)$$

Exercise 2. Show that $(dy_1 + A_{ij} y_i)_e(X_h) = 0$ for all $X_h \in H_e$. *Hint: $X_h \sim [\tilde{\gamma}_h(t)]$ where $\tilde{\gamma}_h(t)$ is a particular horizontal lift.*

Question: How should we interpret $d\mathcal{I}_H \subseteq \mathcal{I}_H$? Examine

$$d(dy_i + A_{ij} y_j) = d(A_{ij} y_j) = (dA_{ij})y_j - A_{ij} \wedge dy_j$$

Is this in $\mathcal{I}_H$? Write $dy_j = (dy_j + A_{jk} y_k) - A_{jk} y_k$.

Exercise 3. Show that $d(dy_i + A_{ij} y_j) = (dA + A \wedge A)y_j$+something in $\mathcal{I}_H$.

Conclusion: If $dA + A \wedge A$ vanishes, then $d\mathcal{I}_H \subseteq \mathcal{I}_H$ and so the horizontal distribution will be integrable.

Definition 1. Denote this form by $F_A = dA + A \wedge A$.

Observe that F_A is a matrix of two forms.

Claim. *If $\{A^\alpha\}$ are connection 1-forms with respect to a local trivialization, $\{U_\alpha\}$ and $F_\alpha = dA^\alpha + A^\alpha \wedge A^\alpha$, then $\{F_\alpha\}$ defines a global section of $\wedge^2 T^*B \otimes End(E)$.*

To establish the claim, suppose that $\{e_i^\alpha\}$ is a local frame over U_α and the transition functions are $\{g^{\beta\alpha}\}$. We need the following diagram to commute:

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "F_\alpha"] \arrow[d, "g^{\beta\alpha}"'] & \mathbb{R}^k \arrow[d, "g^{\beta\alpha}"] \\
\mathbb{R}^k \arrow[r, "F_\beta"'] & \mathbb{R}^k
\end{tikzcd}$$

That is, for $F_\alpha = g^{\alpha\beta} \circ F_\beta \circ g^{\beta\alpha}$.

Exercise 4. Prove this! Hint: Use the transformation law for A^α, e.g,

$$A^\alpha = g^{\alpha\beta} A^\beta g^{\beta\alpha} + g^{\alpha\beta} dg^{\beta\alpha}.$$

Compute dA^α and $A^\alpha \wedge A^\alpha$. [Use the fact that $g^{\alpha\beta} g^{\beta\alpha} = 1$ and so $dg^{\alpha\beta} g^{\beta\alpha} + g^{\alpha\beta} dg^{\beta\alpha} = 0$.]

Definition 2. $F_D = \{F_\alpha\}$ is called the curvature of D.

An alternative desription of the curvature of a connection can be made by $F_D = D^2$. However, we need to make sense of D^2, now. That is, we need to define $D : \Omega^p(B, E) \to \Omega^{p+1}(B, E)$.

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu