THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 22

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Class Notes From Math 433

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Induced Connections II

(3) Induced Connections on E^*:

Given a connection D on $E \to B$, we can define a connection D^* on E^* such that if $s^* \in \Omega^0(E^*)$ and $t \in \Omega^0(E)$, then $[s^*(t)]_b = s_b^*(t_b), \forall b \in B$, is a smooth funtion on the base B. So we can take the differential

$$ds^*(t) = (D^* s^*)(t) + s^*(D(t)).$$

With respect to the local frame $\{e_i\}$ for E, $D = d + A$ and with respect to the dual frame $\{e_i^*\}$ for E^*, $D^* = d + A^*$. Then we have

$$e_i^*(b)(e_j(b)) = \delta_{ij},$$

such that (i) $de_i^*(e_j) = 0$, (ii) $De_i = A_{ji} e_k$, and (iii) $D^* e_j^* = A_{kj}^* e_k^*$. So we need

$$\begin{aligned} 0 &= de_i^*(e_j) \\ &= (A_{ki}^* \otimes e_k^*)(e_j) + e_i^* \otimes (A_{kj} e_k) \\ &= A_{ji}^* + A_{ij}. \end{aligned}$$

Therefore we require

$$A^* = -A^t.$$

Exercise 1. Check that if $\{A^\alpha\}$ is a collection of connection 1-forms for D, then $\{-(A^\alpha)^t\}$ defines a connection on E^*! (If so, then the connection clearly is the one we need!).

Exercise 2. Describe this in terms of horizontal lifting of curves.

Note. We can combine (2) and (3) to get D on $(\overset{r}{\bigotimes} E_1) \otimes (\overset{s}{\bigotimes} E_2)$ etc.

Remark. From (1), (2), and (3), we can obtain the connection on $E_1 \otimes E_2^* \cong \mathrm{Hom}(E_2, E_1)$.

(4) Connections on $\mathrm{Hom}(E_1, E_2) \cong E_2 \otimes E_1^*$:

Since $\mathrm{Hom}(E_1, E_2) \cong E_2 \otimes E_1^*$, so we have the connection

$$D = D_2 \otimes I_1 + I_2 \otimes D_1^*$$

defined on $\mathrm{Hom}(E_1, E_2)$.

Direct description on $\mathrm{Hom}(E_1, E_2)$: Given $h : E_1 \to E_2$ and fix bases $\{e_i^{(1)}\}$ and $\{e_i^{(2)}\}$ for E_1 and E_2, respectively. Then $h = [h_{ij}]$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Question. What is the $D(h)$?

Let $e_{ij} = e_i^{(2)} \otimes e_j^{(1)^*}$. Then

$$\begin{aligned} D(e_{ij}) &= A_{ki}^{(2)} e_k^{(2)} \otimes e_j^{(1)^*} + e_i^{(1)} \otimes A_{kj}^{(1)^*} e_k^{(1)^*} \\ &= A_{ki}^{(2)} e_k^{(2)} \otimes e_j^{(1)^*} - A_{jk}^{(1)} e_i^{(2)} \otimes e_k^{(1)^*}. \end{aligned}$$

But $h = \sum h_{ij}(e_i^{(2)} \otimes e_j^{(1)^*})$, where we think of $\{e_{ij}\}$ as a basis for $\mathrm{Hom}(E_1, E_2)$ via $(e_i^{(2)} \otimes e_j^{(1)^*})(s) = e_j^{(1)^*}(s) e_i^{(1)}$.

Exercise 3. Show that

$$D(h) = \sum (dh + A^{(2)}h - hA^{(1)})_{ij}\, e_i^{(2)} \otimes e_j^{(1)^*}.$$

I.e. if $h = [h_{ij}]$ with respect to $\{e_{ij}\}$, then

$$D(h) = dh + A^{(2)}h - hA^{(1)}.$$

Remark. Special case: if $E_1 = E_2 = E$, then $\mathrm{Hom}(E_1, E_2) = \mathrm{End}(E)$ and D on E induces D on $\mathrm{End}(E)$ such that with respect to local frames, if $D = d + A$ on E and $u \in \Omega^0(\mathrm{End}(E))$, then

$$\begin{aligned} D(u) &= du + Au - uA \\ &= du + [A, u] \end{aligned}$$

on $\mathrm{End}(E)$.

Note. We can use the extension of D on $\mathrm{End}(E)$ to define the *Covariant Derivative*

$$D : \Omega^p(\mathrm{End}(E)) \to \Omega^{p+1}(\mathrm{End}(E))$$

Apply this covariant derivative to $F_D \in \Omega^2(\mathrm{End}(E))$ to compute $D(F_D)$. With respect to local frames, $D(F_A) = dF_A + [A, F_A]$, where $D = d + A$. Thus the Bianchi identity says $D(F_D) = 0$!

(5)<u>Connections on Pull-Back Bundles</u>:

Connection on $f^*(E)$ via

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

Say D is a connection on E and $\{e_i^\alpha\}$ is local frames for E over U_α. Let $\{f^*(e_i^\alpha)\}$ be the "pull-back" frame for $f^*(E)$ over $f^{-1}(U_\alpha)$ with $f^*(e_i^\alpha)(x) = e_i^\alpha(f(x))$. Say $D = d + A^\alpha$ with respect to $\{e_i^\alpha\}$.

Claim. *$f^*(A^\alpha)$ defines connection 1-form on $f^{-1}(U_\alpha)$. (w.r.t. $\{f^*(e_i^\alpha)\}$).*

proof: Given a vector field Y on $f^{-1}(U_\alpha)(\in X)$, $f^*(A_x^\alpha)(Y) = A_{f(x)}^\alpha(f_*Y)$. That is, we define $f^*(D)$ on $f^*(E)$ such that

$$f^*(D_Y)(f^*s) = D_{f \circ Y}(s).$$

□

Note. Given an isomorphic bundle map (h is an isomorphism between two bundles)

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr] & & F \arrow[dl] \\ & B & \end{tikzcd}$$

Let D be a connection on F. Then we can define $h^*(D)$ on E by $h^*(D)(s) = h^{-1}(D)(h(s))$. Therefore

$$h^*(D) = h^{-1} \circ D \circ h.$$

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