THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 24

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Recall that given a bundle, $E \to B$, a bundle metric, $\langle \quad , \quad \rangle$, and a connection D, we say that the metric is compatible with the connection if for all $s, t \in \Omega^0(E)$ the following product formula holds

$$d\langle s, t\rangle = \langle Ds, t\rangle + \langle s, Dt\rangle$$

Such a connection is sometimes called an *orthogonal* connection.

If E is a complex bundle and $\langle \quad , \quad \rangle$ is a Hermitian metric, then the connections which satisfy the above compatibility condition are called *unitary* connections.

Suppose that $\{e_i\}$ is a local frame for E with $\langle e_i, e_j\rangle = \delta_{ij}$ and that with respect to this choice of a local frame, $D = d + A$. Then, we have that

$$\begin{aligned} 0 = d\langle e_i, e_j\rangle &= \langle A_{ki} \otimes e_k, e_j\rangle + \langle e_i, A_{kj} \otimes e_k\rangle \\ &= A_{ki}\delta_{ki} + A_{kj}\delta_{kj} \end{aligned}$$

Hence, $A_{ji} + A_{ij} = 0$.

Note. In the complex case, using a Hermitian metric we get $A_{ji} + \bar{A}_{ij} = 0$.

This says that for an orthogonal connection, the connection 1-form, A, is skew symmetric with respect to an orthogonal frame. (If $\mathrm{Lie}(\mathrm{O}(n))$ denotes the Lie algebra on $\mathrm{O}(n)$, then we can identify the skew symmetric matrices with $\mathrm{Lie}(\mathrm{O}(n))$).

Similarly, for unitary connections on complex bundles, the connection 1-forms take their values in $\mathrm{Lie}(\mathrm{U}(n))$ when computed in a unitary frame.

Note. We can realize E as an associated principal $\mathrm{O}(k)$-bundle. This turns out to be a general feature of connections on principal $G-$bundles: The connection 1-forms take their values in $\mathrm{Lie}(G)$.

A direct computation show that $F_A = dA + A \wedge A$ satisfies $F_A + F_A^t = 0$, again with respect to an orthogonal frame. Thus for any vector fields $X, Y \in TB$, the bundle map $F(X, Y) : E \to E$ is an isometry, i.e, with respect to orthogonal frames, $F(X, Y)$ is $\mathrm{O}(n)$-valued.

An Alternative Description of the Compatibility Condition

A bundle metric, $\langle \quad , \quad \rangle$, can be viewed as a section of $E^* \otimes E^*$. On a vector space, $\langle \quad , \quad \rangle$ is a bilinear map, $V \times V \to \mathbb{R}$, which is equivalent to a map $V \to V^*$. Thus, on E, $\langle \quad , \quad \rangle$ is equivalent to a map, $E \to E^*$; that is, to a section of $\mathrm{Hom}(E, E^*) \cong E^* \otimes E^*$.

Typeset by $\mathcal{AMS}$-TEX

Given a local frame, $\{e_i\}$ of E, we define

$$H = \sum H_{ij} e_i^* \otimes e_j^*$$

where $H_{ij} = \langle e_i, e_j \rangle$. So, $H(s,t) = \langle s, t \rangle$. But, given a connection, D, on E, we get an induced connection on $E^* \otimes E^*$, denoted also by D. So, we can evaluate $D(H)$.

Exercise 1. Show that the following are equivalent:
(1) $d\langle s,t \rangle = \langle Ds, t \rangle + \langle s, Dt \rangle$.
(2) $D(H) = 0$.

Condition 2 above is sometimes called a covariant constant condition. That is, called H covariant constant if $D(H) = 0$.

Connections on TM - The Tangent Bundle

Recall that $TM \to M$ is a vector bundle with rank the dimension of M. We can define connections,

$$D : \Omega^0(TM) \to \Omega^0(T^*M \otimes TM) = \Omega^1(TM)$$

Sections of TM are just vector fields. In this case, $D_V(s)$ is the covariant derivative along the vector field V of the section s. But, V and s are now the same sort of creature: Vector Fields. When speaking of connections on a tangent bundle, we use the symbol ∇, rather than D. As before, $F_\nabla \in \Omega^2(\mathrm{End}(TM))$ and given $X, Y \in \Omega^0(TM)$, we get $F_\nabla(X,Y) : TM \to TM$.

Recall that given a frame, $\{e_i\}$ and a connection $D = d + A$ with respect to this frame, A is a matrix of 1-forms. Write

$$A = \sum_{\alpha=1}^{n} A_\alpha dx_\alpha$$

with respect to the coordinates $(x_1, \ldots, x_n)$, where A_α is a matrix (map on the fibers of E). For $E = TM$, we can take

$$\{\frac{\partial}{\partial x_i}\}_{i=1}^m,$$

where m is the dimension of M, as a local frame for TM over the coordinate patch where the coordinates are $(x_1, \ldots, x_n)$. If $\nabla = d + A$, then

$$\nabla(\frac{\partial}{\partial x_i}) = A_{ji}(\frac{\partial}{\partial x_i}) = A_{ji,k} dx_k \otimes \frac{\partial}{\partial x_j}$$

This is usually written $\Gamma^k_{ji} dx_k \otimes \frac{\partial}{\partial x_j}$. The Γ^k_{ji} are called the *Cristofell* symbols. The curvature of ∇ is usually written, for maximal confusion, as Ω.

Parallel Transport and Horizontal Lifting

Given a curve, $\gamma(t)$, in M, a lift to TM is a vector field along $\gamma(t)$. Given such a lift, says $\tilde{\gamma}(t)$, we can compute the covariant derivative with respect to $\dot{\gamma}(t)$ along $\tilde{\gamma}(t)$, $\nabla_{\dot{\gamma}(t)}\tilde{\gamma}(t)$. But, for our lift, $\tilde{\gamma}(t)$, we can simply take $\dot{\gamma}(t)$.

Definition 1. Say that $\gamma(t)$ is *geodesic* if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$.

With respect to local coordinates, $(x_1, \ldots, x_m)$, a curve $\gamma(t)$ has component functions, $(x_1(t), \ldots, x_m(t))$. Since $\nabla = d + \Gamma_{ij}^k dx_k$, we have the following exercise.

Exercise 2. Show that the geodesic condition yields a system of ordinary differential equations,

$$\ddot{x}_k(t) + \Gamma_{ij}^k \gamma(t)\dot{x}_i(t)\dot{x}_j(t) = 0$$

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