THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 3

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Recall that a bundle is a quadruple, (E, B, F, π), where E, B, F are topological spaces, $\pi : E \to B$ is a continuous map satisfying $\pi^{-1}(x) \cong F$ for all $x \in B$ and there exists an open set $U \subseteq B$ with $x \in U$ such that $\pi^{-1}(U) \cong U \times F$ in a fiber preserving way. In the future, the bundle will just get denoted, $\pi : E \to B$. If we are in the **Smooth** category, π will be a smooth map.

Definition 1. For $\pi : E \to B$ a bundle, we say that a map $s : B \to E$ is a section if $\pi \circ s = \mathrm{Id}_B$. That is, $s(b)$ is in the fiber of s over B.

A special case of a section is given by the trivial bundle,

$$\begin{tikzcd} E = B \times F \arrow[d, "\pi=\text{projection}"] \\ B \end{tikzcd}$$

If s is a section, then $s(b) = (b, \sigma(b))$, where $\sigma : B \to F$. Conversely, if $\sigma : B \to F$, then the map $s : B \to B \times F$ given by $s(b) = (b, \sigma(b))$ is a section. That is, there is a 1-1 correspondance between sections of the bundle and $Map(B, F)$.

When E is not the trivial bundle, it will help to think of sections as a sort of twisted map from B to F. Write $E_{|U}$ for $\pi^{-1}(U)$. If $\psi : E_{|U} \to U \times F$ is an identification, then we get the following diagram

$$\begin{tikzcd} E|U \arrow[rr, "\psi"] & & U \times F \\ & U \arrow[ul, "s"] \arrow[ur, "\psi \circ s = s_U"'] & \end{tikzcd}$$

Using the local trivialization, the local desription of the section is a map $U \to F$.

Question: Do sections always exist?

Answer:

(a) For smooth vector bundles the answer is yes. (C^∞ sections).

(b) For holomorphic bundles the answer is no.

(c) For principal bundles the answer is no.

We have defined the objects which we want to study, and now we defined the maps between them.

Definition 2. If $\pi : E \to B$ and $\pi' : E' \to B'$ are two bundles, a bundle map (or map of bundles) is a pair of maps, (u, f) such that the diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\mathbf{u}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "\mathbf{f}"'] & B' \end{tikzcd}$$

The definition says that $E_{|b} \mapsto E'_{|f(b)}$ under u. If $E = E'$and $B = B'$, then (u, f) is called a bundle endomorphism. If the maps are invertible, then (u, f) is a bundle automorphism. Observe that if we are considering vector bundles, then on the fibers the maps must be linear. If we are considering principal G-bundles, then we require the map on the fibers to group homomorphism and be G equivariant. That is, $f(p \cdot g) = f(p) \cdot g$.

Note the special case of a bundle endomorphism where $f = \mathrm{Id}_B$. Then, we have that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\mathbf{u}"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

So, u just transforms points in each fiber. For the case of a vector bundle, on each fiber u is a vector space isomorphism.

Local Picture for Bundles

Suppose that $\pi : E \to B$ is a smooth vector bundle with $\pi^{-1}(b) \cong \mathbb{R}^n$. For each $b \in B$, take a neighborhood, say U_b, over which E can be trivialized. The collection $\{U_b\}_{b \in B}$ then covers B. Let $\psi_b : E_{|U_b} \to U_b \times \mathbb{R}^n$ be the trivialization. Suppose that $b \in U_\alpha \cap U_\beta$. A natural question to ask is how ψ_α and ψ_β compare. We have the following diagram:

$$\begin{tikzcd} & E|U_\alpha \cap U_\beta \arrow[dl, "\psi_\beta"'] \arrow[dr, "\psi_\alpha"] & \\ (U_\alpha \cap U_\beta) \times \mathbb{R}^n & & (U_\alpha \cap U_\beta) \times \mathbb{R}^n \arrow[ll, "g_{\beta\alpha}"] \end{tikzcd}$$

Where $g_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}$. Observe that $g_{\beta\alpha}$, being the composite of linear maps, is itself a linear map. Since ψ_α and ψ_β are invertible, so is $g_{\beta\alpha}$. That is we may regard $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$. We get one of these for every pair, $U_\alpha \cap U_\beta \neq \emptyset$.

So, given a smooth vector bundle, we obtain a cover $\{U_\alpha\}$ of B by locally trivial neighborhoods and a set of transition functions,

$$\{g_{\beta\alpha}\}_{U_\alpha \cap U_\beta \neq \emptyset}$$

<u>Question:</u> When are the cover and the transition functions equivalent to the information contained in the bundle? That is, when does $\{U_\alpha\}$ and $\{g_{\beta\alpha}\}$ describe a bundle?

One observation can be made right away. If the cover and the transition functions do determine a cover, then we must have $g_{\alpha\alpha} = \mathrm{Id}$, the inverse of $g_{\beta\alpha}$ is $g_{\alpha\beta}$, and if $U_\alpha \cap U_beta \cap U_\gamma \neq \emptyset$, $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{Id}$ This last condition is known as the cocycle condition.

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