THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 31

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Class Notes From Math 433

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Characteristic Classes for Complex Bundles III

We saw last time: for complex bundles $E_i \to B$, $i = 1, 2$,

$$
\begin{aligned}
c(E_1 \oplus E_2) &= c(E_1) \cdot c(E_2) \\
ch(E_1 \oplus E_2) &= ch(E_1) + ch(E_2)
\end{aligned}
$$

where $c(E) = \left[\det(I + \frac{i}{2\pi}F)\right]$ (F is a curvature of any connection) and $ch(E) = \left[\mathrm{Tr}(\exp(\frac{i}{2\pi}F))\right]$

Chern Classes for Dual Bundles

Say E^*= dual bundle to E. Recall: given a connection D on E, we can define a connection D^* on E^* such that $A^* = -A^t$. Thus

$$
\begin{aligned}
F^* &= dA^* + A^* \wedge A^* \\
&= -dA^t + A^t \wedge A^t.
\end{aligned}
$$

But $A^t \wedge A^t = -(A \wedge A)^t$, if A is a matrix of 1 forms. So the curvature $F^* = -F^t$. Then

$$
\begin{aligned}
c(E^*) &= \det(I - \frac{i}{2\pi}F)^t \\
&= \det(I - \frac{i}{2\pi}F) \\
&= \sum_k c_k(E^*) \\
&= \sum_k c_k(-\frac{i}{2\pi}F) \\
&= \sum_k (-1)^k c_k(\frac{i}{2\pi}F).
\end{aligned}
$$

So

$$c_k(E^*) = (-1)^k c_k(E).$$

For example, $c_1(E^*) = -c_1(E)$, etc.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Relations between Chern Classes c_k and Chern Characters ch_k

We know $ch_k(A) = \mathrm{Tr}(A^k)$, so

$$\begin{aligned} ch_0(E) &= \mathrm{Tr}(\frac{i}{2\pi}F)^0 = \mathrm{Tr}\, I = \mathrm{Rank}\, E \\ ch_1(E) &= \mathrm{Tr}(\frac{i}{2\pi}F)^1 = c_1(E) \\ ch_2(E) &= \mathrm{Tr}(\frac{i}{2\pi}F)^2 \\ &\vdots \end{aligned}$$

If the eigenvalues of $\frac{i}{2\pi}F$ are $x_1, \cdots, x_n$, then

$$\begin{aligned} ch_2(E) &= \frac{1}{2}\sum_{i=1}^{n} x_i^2 \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{i<j} x_i x_j\right) \\ &= \frac{1}{2}c_1(E)^2 - c_2(E) \end{aligned}$$

Thus

$$ch_2(E) = \frac{1}{2}c_1(E)^2 - c_2(E)$$

etc.

Chern Characters as a Map between Bundles and Cohomology

Property. $ch(E_1 \otimes E_2) = ch(E_1) \cdot ch(E_2)$.

Hint: Given connections D_i on E_i, $i = 1, 2$, we can define $D = D_1 \otimes I_2 + I_1 \otimes D_2$ with $F_d = F_1 \otimes I_2 + I_1 \otimes F_2$ on $E_1 \otimes E_2$. Then use $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}\, A \cdot \mathrm{Tr}\, B$. □

Exercise 1. Prove the above claim.

Remark. Given B, if we define $\mathrm{Vect}(B) := \{$Isomorphism classes of complex bundle on $B\}$, then we have two "ring operations" on $\mathrm{Vect}(B)$:

$$\begin{aligned} + &: \quad E_1 \oplus E_2 \\ \cdot &: \quad E_1 \otimes E_2. \end{aligned}$$

$\mathrm{Vect}(B)$ can be made into a ring under these operations (cf. a course on K-theory!) in which case

$$ch : \mathrm{Vect}(B) \longrightarrow H^*(B, \mathbb{R})$$

defines a ring isomorphism.

Note. For a complex line bundle $L \to B$, we only have

$$\begin{aligned} c_0(L) &= 1 \\ c_1(L) &\in H^2(B, \mathbb{R}). \end{aligned}$$

(will see later how to compute in the case of L over complex manifold!)

So the total Chern class

$$c(L) = (1 + c_1(L)).$$

For $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ (sum of n line bundles), we can get

$$\begin{aligned} c(E) &= c(L_1)c(L_2)\cdots c(L_n) \\ &= \prod_{i=1}^{n}(1 + c_1(L_i)) \end{aligned}$$

For example,

$$\begin{aligned} c_1(E) &= \sum_{i=1}^{n} c_1(L_1) \\ c_2(E) &= \sum_{i\neq j} c_1(L_i)c_1(L_j) \\ &\vdots \\ c_n(E) &= \prod_{i=1}^{n} c_1(L_i) \end{aligned}$$

Exercise 2. Suppose $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. If $c_1(L_i) = 0$, for some i, then $c_n(E) = 0$

Exercise 3. Suppose $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$. If $c_1(L_1) = c_1(L_2) = \cdots = c_1(L_k) = 0$, then $c_j(E) = 0$, for $j > n - k$.

Characteristic Classes for Real Bundles I

We want to define the *characteristic classes* for real bundles, say $E_{\mathbb{R}} \to B$ of rank n ($E_{\mathbb{R}}|_b \cong \mathbb{R}^n$).

First Approach: Complexify $E_{\mathbb{R}}$ and use the previous discussions about complex bundles.

Let's recall some facts. We complexify a real vector space $V_{\mathbb{R}}$ by taking $V_{\mathbb{R}} \otimes \mathbb{C} \equiv V_{\mathbb{C}}$. If $\dim_{\mathbb{R}}(V_{\mathbb{R}}) = n$, then $\dim_{\mathbb{C}}(V_{\mathbb{C}}) = n$. (If $V_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\{v_1, \cdots, v_n\}$, then $V_{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\{v_1, \cdots, v_n\}$.)

For a real bundle $E_{\mathbb{R}}$ of rank n, we can form a *complex bundle* $E_{\mathbb{C}} := E_{\mathbb{R}} \otimes \mathbb{C}$ as follows:

If

$$E_{\mathbb{R}} = \coprod_{\alpha} U_\alpha \times \mathbb{R}^n / g_{\alpha\beta} \qquad \text{where} \quad g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R}),$$

then

$$E_{\mathbb{C}} = E_{\mathbb{R}} \otimes \mathbb{C} = \coprod_{\alpha} U_\alpha \times \mathbb{C}^n / g_{\alpha\beta} \qquad \text{where} \quad g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R}) \hookrightarrow \mathrm{GL}(n, \mathbb{C}).$$

Therefore we can define the **characteristic classes** for $E_{\mathbb{R}}$ by

$$P(E_{\mathbb{R}}) \equiv P(E_{\mathbb{C}}) \qquad \text{for any } \mathrm{GL}(n, \mathbb{C})\text{-invariant polynomial } P.$$

For example, $c_k^{\mathbb{R}}(E_{\mathbb{R}}) \equiv c_k(E_{\mathbb{C}})$.

Note. We will see that for k odd, $c_k(E_{\mathbb{R}}) = 0$.

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