THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 36

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CLASS NOTES FROM MATH 433

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Euler Classes for Orientable Vector Bundles

Behaviour of $e(E)$ under Change in Orientation

Say $E \to B$ is rank $n = 2m$ real oriented bundle with

$$E = \coprod U_\alpha \times \mathbb{R}^n / g_{\alpha\beta} \qquad g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n)\,.$$

So

$$E|_{U_\alpha} \xrightarrow[\simeq]{\psi_\alpha} U_\alpha \times \mathbb{R}^n$$

defines *oriented* frames.

<u>To change orientation</u>: Pick $g_0 \in \mathrm{O}(n) \backslash \mathrm{SO}(n)$ $(\det(g_0) = -1)$. Define new set of frames

$$\widetilde{\psi_\alpha} : E|_{U_\alpha} \xrightarrow[\simeq]{\psi_\alpha} U_\alpha \times \mathbb{R}^n \xrightarrow{1\times g_0} U_\alpha \times \mathbb{R}^n.$$

Then new transition functions are $\widetilde{g}_{\alpha\beta} = g_0 \circ g_{\beta\alpha} \circ g_0^{-1}$

$$\begin{tikzcd}
E|_{U_\alpha\cap U_\beta} \arrow[r, "\psi_\alpha"] \arrow[dr, "\psi_\beta"'] & \mathbb{R}^n \arrow[r, "g_0"] \arrow[d, "g_{\beta\alpha}"] & \mathbb{R}^n \arrow[d, "\widetilde{g}_{\alpha\beta}=g_0\circ g_{\beta\alpha}\circ g_0^{-1}"] \\
& \mathbb{R}^n \arrow[r, "g_0"'] & \mathbb{R}^n
\end{tikzcd}$$

Note.

(1) $\det(\widetilde{g}_{\beta\alpha}) = +1$, so $\widetilde{g}_{\beta\alpha} \in \mathrm{SO}(n)$.

(2) Using $\widetilde{\psi_\alpha}$ to define an orientation on fibers yields *opposite* orientation to that obtained via ψ_α.

Exercise 1. Show that if F_α is a curvature for a connection on E with respect to oriented local frames and $\widetilde{F_\alpha}$ is a corresponding curvature on $\widetilde{E}$, then

$$\widetilde{F_\alpha} = g_0 \circ F_\alpha \circ g_0^{-1}.$$

Thus

$$\begin{aligned} e(\widetilde{E}) &= e(\frac{1}{2\pi}\widetilde{F_\alpha}) \\ &= e(g_0(\frac{1}{2\pi}F_\alpha)g_0^{-1}) \\ &= -e(\frac{1}{2\pi}F_\alpha) \\ &= -e(E)\,. \end{aligned}$$

Special Case: $E = TM$ **Tangent Bundle of** M

Definition 1. (1) M is *orientable* $\Longleftrightarrow$ TM is orientable. (2) The *Euler class* of an orientable manifold M is defined to be $e(M) \equiv e(TM)$.

Remark. Suppose that M is an orientable manifold of dimension $2m$, and that $e(M) \in H^{2m}(M, \mathbb{Z})$ is its Euler class. We can evalute $e(M)$ on the fundamental class in $H_{2m}(M, \mathbb{Z})$, i.e., on the class $[M] \in H_{2m}(M, \mathbb{Z})$.

It is an important fact that

$$\begin{aligned} e(M)[M] &= \int_M e(M) \\ &= \chi(M) && \text{Euler characteristic} \\ &= \sum_{k=0}^{2m} (-1)^k b_k(M) && b_k(M) = \text{Betti numbers}\,. \end{aligned}$$

This is an example of an **Index Theorem**.

Example. Special Case: $\dim(M) = 2$

A skew symmetric 2×2 matrix is of the form

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}.$$

Thus (cf. the definition of e in Lecture 35) $e(A) = (ae_1 \wedge e_2\,,\, e_1 \wedge e_2) = a$, where $\{e_1, e_2\}$ is an oriented basis for $\mathbb{R}^2$. Applying this to the curvature of a connection, we get $e(TM) = e(\frac{1}{2\pi}\Omega)$, where Ω is a curvature of Levi-Civita connection. But Ω can be written

$$\Omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},$$

where ω is a 2 form and hence a multiple of the area form on M.

Writing $\omega = \kappa dA$, we can see that $e(TM) = \frac{1}{2\pi}\kappa dA$ and hence $\int_M e(TM) = \frac{1}{2\pi}\int_M \kappa\, dA$. Therefore

$$(*) \qquad \chi(M) = \frac{1}{2\pi}\int_M \kappa\, dA\,.$$

The function κ is the Gauss curvature of the metric on M and $(*)$ is the **Gauss-Bonet Theorem**.

How to Detect Orientability

Suppose

$$E = \coprod U_\alpha \times \mathbb{R}^n / g_{\alpha\beta}$$

is a vector bundle with

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(n)\,.$$

Question: Can we change the local trvializations so that the new transition functions $\widetilde{g}_{\alpha\beta}$ are in $\mathrm{SO}(n)$?

To do so, we need

$$\lambda_\alpha : U_\alpha \to \mathrm{O}(n)$$

such that if $\widetilde{\psi_\alpha}$ defined by $E|_{U_\alpha} \xrightarrow[\simeq]{\psi_\alpha} U_\alpha \times \mathbb{R}^n \xrightarrow{\lambda_\alpha} U_\alpha \times \mathbb{R}^n$, then

$$\widetilde{g}_{\beta\alpha} = \lambda_\beta \circ g_{\beta\alpha} \circ \lambda_\alpha^{-1} \ \in \ \mathrm{SO}(n)\,.$$

The following diagram shows that $\widetilde{g}_{\alpha\beta}$ are indeed the new transition functions.

$$\begin{tikzcd}
E|_{U_\alpha \cap U_\beta} \arrow[r, "\psi_\alpha"] \arrow[dr, "\psi_\beta"'] & \mathbb{R}^n \arrow[r, "\lambda_\alpha"] \arrow[d, "g_{\beta\alpha}"] & \mathbb{R}^n \arrow[d, "\widetilde{g}_{\beta\alpha} = \lambda_\beta \circ g_{\beta\alpha} \circ \lambda_\alpha^{-1}"] \\
 & \mathbb{R}^n \arrow[r, "\lambda_\beta"'] & \mathbb{R}^n
\end{tikzcd}$$

Defining

$$\begin{cases} \delta_{\alpha\beta} = \det(g_{\alpha\beta}) : U_\alpha \cap U_\beta \to \{\pm 1\} = \mathbb{Z}_2 \\ l_\alpha = \det(\lambda_\alpha) : U_\alpha \to \mathbb{Z}_2\,, \end{cases}$$

we see that the $\{\lambda_\alpha\}$ must be such that

$$1 = \widetilde{\delta}_{\alpha\beta} = \det(\widetilde{g}_{\alpha\beta}) = \det(\lambda_\alpha \circ g_{\alpha\beta} \circ \lambda_\beta^{-1}) = l_\alpha \cdot \delta_{\alpha\beta} \cdot l_\beta^{-1}\ .$$

That is, we require

$$\delta_{\alpha\beta} = \frac{l_\beta}{l_\alpha}\,.$$

Reformulation:

Writing

$$\begin{aligned} \delta_{\alpha\beta} &= e^{i\pi u_{\alpha\beta}} \\ l_\alpha &= e^{i\pi v_\alpha} \end{aligned} \qquad\qquad u_{\alpha\beta}\ ,\ v_\alpha \in \{0,1\} = \mathbb{Z}_2\,,$$

the question becomes

Question: Given $\{u_{\alpha\beta}\}$, can we find $\{v_\alpha\}$ such that

$$(**) \qquad\qquad u_{\alpha\beta} = v_\beta - v_\alpha\ ?$$

Note. We are only interested in $(**)$ for $\{u_{\alpha\beta}\}$ which come from transition functions, i.e., which satisfy

$$I = g_{\alpha\beta}\ g_{\beta\gamma}\ g_{\gamma\alpha} \qquad \text{on} \qquad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset\,,$$

that is, which satisfy

$$1 = \delta_{\alpha\beta}\ \delta_{\beta\gamma}\ \delta_{\gamma\alpha}$$

or

$$0 = u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha}.$$

Question becomes: Given $\{u_{\alpha\beta}\}$ such that

$$u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = 0 \qquad \text{on} \qquad U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset\,, \tag{1}$$

can we find v_α such that

$$u_{\alpha\beta} = v_\beta - v_\alpha \ ? \tag{2}$$

Notice that (1) is the closed condition on Čech 1-cochains and (2) is the exact condition.

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