THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY
LECTURE 38

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Last time: We saw that the obstruction to orientability of a vector bundle $E \to B$ is the first Stiefel-Whitney class $w_1(E) \in H^2(B, \mathbb{Z}_2)$.

Suppose that $w_1(E) = 0$, so that we can write $E = \coprod U_\alpha \times \mathbb{R}^n / g_{\alpha\beta}$, $g_{\alpha\beta} \in \underline{\mathrm{SO}(n)}$.

In this case, we can look for a more subtle structure called a **spin structure**.

Spin Structures

Replace E by principal $\mathrm{SO}(n)$-bundle:

$$P_{\mathrm{SO}(n)} = \coprod U_\alpha \times \mathrm{SO}(n)/g_{\alpha\beta} \ .$$

Key Facts:

- For $n > 2$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$.
- There exists a simply connected Lie group $\mathrm{Spin}(n)$ such that

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \to 1 \ ,$$

  that is, ρ is a *double cover* and *group homomorphism.*

Example 1. $\mathrm{Spin}(2) = S^1$. ($S^1 \to S^1$ via $z \mapsto z^2$.)

Example 2. $\mathrm{Spin}(3) = \mathrm{SU}(2) = 2 \times 2$ unitary complex matrices with determinant 1.

To see $\mathrm{Spin}(3) \to \mathrm{SO}(3)$:

- $A \in \mathrm{SU}(2)$ can be written as

$$A = \begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix}$$

  with $|z_1|^2 + |z_2|^2 = 1$. So

$$\begin{aligned} \mathrm{SU}(2) &\cong \{ \ (z_1, z_1) \ : \ |z_1|^2 + |z_2|^2 = 1 \ \} \cong S^3 \\ &\cong \{ \ q = z_1 + j z_2 \in \mathbb{H} \ : \ |q|^2 = 1 \ \} \ , \end{aligned} \tag{1}$$

  under $\mathbb{C}^2 \cong \mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$ and $q = z_1 + j z_2$.

- Define

$$\begin{aligned} \mathrm{SU}(2) \times \mathbb{H} &\to \mathbb{H} \\ (q_0 \, , q) &\mapsto q_0 \, q \, q_0^{-1} \end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let $\mathbb{R}^3 = \operatorname{Im}\mathbb{H} = i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$. Then the unit quaternians act on $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$ by

$$
\begin{aligned}
\mathrm{SU}(2) \times \operatorname{Im}\mathbb{H} &\to \operatorname{Im}\mathbb{H} \\
(q_0\,, x) \quad &\mapsto q_0\, x\, q_0^{-1} (= q_0\, x\, \overline{q}_0 \quad \text{if } |q_0|^2 = 1\,)
\end{aligned}
\tag{2}
$$

This defines an action of $\mathrm{SU}(2)$ on $\mathbb{R}^3$. In fact, if $|q_0| = 1$ and $x \in \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$, then $x \overset{A_{q_0}}{\mapsto} q_0\, x\, q_0^{-1}$ is an $\mathrm{SO}(3)$ transformation.

Claim:

$$
\begin{aligned}
\mathrm{SU}(2) &\to \mathrm{SO}(3) \\
q_0 &\mapsto A_{q_0}
\end{aligned}
$$

is the $2:1$ double cover ($A_{q_0} = A_{-q_0}$). Hence the map $q_0 \mapsto A_{q_0}$ is the map we want !

Example 3. $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Remark. Given $\mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n)$ and $g \in \mathrm{SO}(n)$, we can always find $\widetilde{g} \in \mathrm{Spin}(n)$ such that $\rho(\widetilde{g}) = g$.

How about

$$
\begin{tikzcd}
 & \mathrm{Spin}(n) \arrow[d, "\rho"] & ? \\
U \arrow[ur, "?\ \exists\ \widetilde{g}"] \arrow[r, "g"] & \mathrm{SO}(n) &
\end{tikzcd}
$$

If U is contractible, then we can do it !

Apply this to $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n)$ in the given principal bundle $P_{\mathrm{SO}(n)} = \coprod U_\alpha \times \mathrm{SO}(n)/g_{\alpha\beta}$ (assuming $\{U_{\alpha\beta}\}$ is “nice”). Thus we can lift $g_{\alpha\beta}$ to $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(n)$ if $U_\alpha \cap U_\beta$ is contractible. That is, the diagram commutes:

$$
\begin{tikzcd}
 & \mathrm{Spin}(n) \arrow[d, "\rho"] \\
U_\alpha \cap U_\beta \arrow[ur, "\widetilde{g}_{\alpha\beta}"] \arrow[r, "g_{\alpha\beta}"'] \arrow[ur, phantom, shift right=3ex, "\circ"] & \mathrm{SO}(n)
\end{tikzcd}
$$

Question: Do the transition functions $\{\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(n)\}$ define a $\mathrm{Spin}(n)$ **bundle**, i.e., do we get

$$
P_{\mathrm{Spin}(n)} = \coprod U_\alpha \times \mathrm{Spin}(n)/\widetilde{g}_{\alpha\beta} \ ?
$$

If **YES**, then such a lift defines a **Spin structure** on ρ and we have $P_{\mathrm{Spin}(n)} \xrightarrow{\rho} P_{\mathrm{SO}(n)}$, $2:1$ on each lift.

We require (the cocycle condition) $\widetilde{g}_{\alpha\beta}\, \widetilde{g}_{\beta\gamma}\, \widetilde{g}_{\gamma\alpha} = I$ (in $\mathrm{Spin}(n)$).

How to Detect Obstruction:

Given $\{g_{\alpha\beta}\}$, we can pick a system of lifts $\{\widetilde{g}_{\alpha\beta}\}$. Define $h_{\alpha\beta\gamma} := \widetilde{g}_{\alpha\beta}\, \widetilde{g}_{\beta\gamma}\, \widetilde{g}_{\gamma\alpha}$ for $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

Note. $h_{\alpha\beta\gamma} \in \mathrm{Spin}(n)$ but $\rho(h_{\alpha\beta\gamma}) = g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = I \in \mathrm{SO}(n)$. So $h_{\alpha\beta\gamma} \in \rho^{-1}(I) \cong \mathbb{Z}^2$

So $h^{(2)} = \{h_{\alpha\beta\gamma}\}$ defines a Čech 2-cochain.

Key Facts. *(With $h_{\alpha\beta\gamma} = e^{i\pi w_{\alpha\beta\gamma}}$ defining $w^{(2)}$.)*

$$\delta w^{(2)} = 0$$

so

$$[w^{(2)}] \in \check{H}^{(2)}(B, \mathbb{Z}_2)$$

is the obstruction !

If $[w^{(2)}] = 0$, then there exists $\{\lambda_{\alpha\beta}\} = \lambda^{(1)}$ a 1-cochain such that $w^{(2)} = \delta(\lambda^{(1)})$.

FACT. *The class $[w^{(2)}] = w_2(P_{\mathrm{SO}(n)}) \in H^2(B, \mathbb{Z}_2)$ is the* **second Stiefel-Whitney class**.

$$P_{\mathrm{SO}(n)} \textit{ admits a spin structure} \Longleftrightarrow w_2(P_{\mathrm{SO}(n)}) = 0\ .$$

Definition 1. An orientable manifold M is *spin* if a principal $\mathrm{SO}(n)$ frame bundle admits a lift to a principal $\mathrm{Spin}(n)$ bundle ($\Longleftrightarrow w_2(TM) \equiv w_2(M) = 0$).

Examples.

(1) S^m is spin.
(2) All orientable Riemann surfaces, Σ^g, are spin.
(3) $\mathbb{CP}(k)$ is spin $\Longleftrightarrow k$ is odd. ($\mathbb{CP}(2)$ is **NOT** spin !)
(4) All Lie groups are spin.

Remark. If M is spin and P_{Spin} is a lift of $\mathrm{SO}(n)$ (bundle of frames), we can get associated vector bundles

$$S = P_{\mathrm{Spin}} \times_\rho V$$

where $\rho : \mathrm{Spin}(n) \to \mathrm{Aut}(V)$ defines **representation of spin ("Spin Bundle")**.

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