THE GEOMETRY OF VECTOR BUNDLES AND AN INTRODUCTION TO GAUGE THEORY LECTURE 9

Professor Steven Bradlow
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University of Illinois at Urbana-Champaign

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New Bundles From Old Bundles II

Given two bundles

$$E = (\coprod U_\alpha \times \mathbb{R}^n)/\{g_{\alpha\beta}\}$$

and

$$F = (\coprod U_\alpha \times \mathbb{R}^n)/\{h_{\alpha\beta}\}.$$

Claim. *There exists a bundle* $\mathrm{Hom}(E, F)$ *such that rank* $= n \cdot m$ *and* $\mathrm{Hom}(E, F)_b = \mathrm{Hom}(E_b, F_b)$.

Proof: There are two points of view.

(1) For vector spaces V and W, $\mathrm{Hom}(V, W) \cong W \otimes V^*$.

Exercise 1. Show that $\mathrm{Hom}(V, W) \cong W \otimes V^*$.

We can define

$$\mathrm{Hom}(E, F) := F \otimes E^*$$

which has the properties: (i) rank nm, (ii) fiber $(F \otimes E^*)_b = F_b \otimes E_b^*$, and (iii) transition functions $h_{\alpha\beta} \otimes {g^t_{\alpha\beta}}^{-1}$.

(2) More directly: We can identify $\mathrm{Hom}(E, F)$ by looking at its sections.

Exercise 2. Show that $s \in \Gamma(\mathrm{Hom}(E, F)) \iff s : E \to F$ is a bundle map.

Using local trivializations, bundle maps can give local descriptions

$$s_\alpha : U_\alpha \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$$

with

$$(*) \qquad s_\beta = h_{\beta\alpha} s_\alpha g^{-1}_{\beta\alpha}.$$

<u>Question:</u> For which bundle is $(*)$ the description of a section?

<u>Answer:</u> Take principal bundles

$$P_F = \coprod U_\alpha \times \mathrm{GL}(m)/\{h_{\alpha\beta}\}$$

and

$$P_E = \coprod U_\alpha \times \mathrm{GL}(n)/\{g_{\alpha\beta}\}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Define

$$P = P_F \times P_E = \coprod U_\alpha \times (\mathrm{GL}(m) \times \mathrm{GL}(n))/\{h_{\alpha\beta} \times g_{\alpha\beta}\}.$$

Take the representation of $\mathrm{GL}(m) \times \mathrm{GL}(n)$ on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ given by

$$\rho : \mathrm{GL}(m) \times \mathrm{GL}(n) \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)),$$

that is, such that $\rho(A, B)(U) = A \circ U \circ B^{-1}$. Then the bundle we want is

$$\mathrm{Hom}(E, F) \equiv P \times_{\mathrm{Ad}} \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

Exercise 3. Show that sections of this are given by $(*)$.

Exercise 4. Show that $\mathrm{Hom}(E, F) \cong E^* \otimes F$!

Remark. The case $E = F$ is an important special case. In this case, $\mathrm{Hom}(E, E) = \mathrm{End}(E)$.

Exercise 5. Show that $\mathrm{End}(E) = P_E \times_{\mathrm{Ad}} \mathrm{Mat}_n$, where

$$\begin{aligned} \mathrm{Ad} : \mathrm{GL}(n) &\longrightarrow \mathrm{GL}(\mathrm{Mat}_n) \\ A &\longmapsto (U \mapsto A \circ U \circ A^{-1}). \end{aligned}$$

Note. If we define $\mathrm{Aut}(E)$ to be the bundle of isomorphisms of E such that $\mathrm{Aut}(E)|_p = \mathrm{Aut}(E_p)$, then $\mathrm{Aut}(E) = P_E \times_{\mathrm{Ad}} \mathrm{GL}(n)$ is a bundle of groups, but not quite a principal bundle (transition functions are not quite by left action of structure group on fibers). If $\mathfrak{G}$ is the space of sections of $\mathrm{Aut}(E)$, then $\mathfrak{G}$ can be given a *group* structure using fiberwise identifications $\mathrm{Aut}(E|_P) \simeq \mathrm{GL}(n)$. The group $\mathfrak{G}$ is the **gauge group** of E.

Kernel, Image, and Cokernel Bundles

Given a bundle map

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & F \arrow[dl] \\ & B & \end{tikzcd}$$

Define

$$(\mathrm{Ker}(f))_b := \mathrm{Ker}(f_b : E_b \to F_b)$$

$$(\mathrm{Im}(f))_b := \mathrm{Im}(f_b : E_b \to F_b)$$

$$(\mathrm{Coker}(f))_b := F_b/(\mathrm{Im}(f)_b)$$

Questions:

(1) Is $\cup_{b\in B}(\mathrm{Ker}(f))_b := \mathrm{Ker}(f)$ a bundle over B?(i.e. a subbundle of E.)

(2) Is $\cup_{b\in B}(\mathrm{Im}(f))_b := \mathrm{Im}(f)$ a subbundle of F?

(3) What about $\cup_{b\in B}(\mathrm{Coker}(f))_b := \mathrm{Coker}(f)$?

Answers: In general, the answers are not **YES**. For example, take trivial bundles:

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^m \arrow[rr, "f"] \arrow[dr] & & \mathbb{R}^n \times \mathbb{R}^m \arrow[dl] \\ & \mathbb{R}^n & \end{tikzcd}$$

and define $f(x) = x \cdot I_n$. (It is certainly a bundle map.) Then

$$(\mathrm{Ker}(f))_x = \begin{cases} 0, & \text{for } x \neq 0 \\ \mathbb{R}^n, & \text{for } x = 0. \end{cases}$$

So fiber dimensions can jump!

Theorem 1. *If* Rank f_b *is* <u>*constant*</u>*, then* $\operatorname{Ker} f$*,* $\operatorname{Im} f$*, and* $\operatorname{Coker} f$ *are bundles.*

Note. The proof uses "Constant Rank Theorem" for bundle maps. It amounts to showing that trivializations for E and F can be chosen so on each U_α,

$$E_b \xrightarrow[\simeq]{\Psi_\alpha} \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k},$$

where $\mathbb{R}^k = \operatorname{Ker}(f_\alpha)$, and

$$F_b \xrightarrow[\simeq]{\Phi_\alpha} \mathbb{R}^m = \mathbb{R}^i \oplus \mathbb{R}^{m-i},$$

where $\mathbb{R}^i = \operatorname{Im}(f_\alpha)$. Then, on $U_\alpha \cap U_\beta$, we have the commutative diagram (w.r.t. given decompositions of $\mathbb{R}^n$):

$$\begin{tikzcd}
\mathbb{R}^k \oplus \mathbb{R}^{n-k} \arrow[r, "f_\alpha"] \arrow[d, "g_{\beta\alpha}"'] & \mathbb{R}^i \oplus \mathbb{R}^{m-i} \arrow[d, "h_{\beta\alpha}"] \\
\mathbb{R}^k \oplus \mathbb{R}^{n-k} \arrow[r, "f_\beta"'] & \mathbb{R}^i \oplus \mathbb{R}^{m-i}
\end{tikzcd}$$

with

$$g_{\beta\alpha} = \begin{pmatrix} k_{\beta\alpha} & \Lambda_{\beta\alpha} \\ 0 & C_{\beta\alpha} \end{pmatrix}$$

$$h_{\beta\alpha} = \begin{pmatrix} i_{\beta\alpha} & \Pi_{\beta\alpha} \\ 0 & D_{\beta\alpha} \end{pmatrix}$$

Note. $\{k_{\beta\alpha}\}$ are transition functions for $\operatorname{Ker} f$. Similarly for $\operatorname{Im} f$ and $\operatorname{Coker} f$.

Exercise 6. Using <u>orthonormal frames</u> (assuming E and F have matrices), we can assume $g_{\beta\alpha} \in \mathrm{O}(n)$ and $k_{\beta\alpha} \in \mathrm{O}(n)$. Show that it follows that $\Lambda_{\beta\alpha} = 0$ and $\Pi_{\beta\alpha} = 0$.

273 Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801
E-mail address: bradlow@math.uiuc.edu